J. DIFFERENTIAL GEOMETRY
46 (1997) 236-286

CONTACT CIRCLES ON 3-MANIFOLDS

HANSJÖRG GEIGES & JESÚS GONZALO

A *contact circle* on a 3-manifold is a pair of contact forms that defines a linear circle of contact forms (see below for the formal definition). This concept was introduced in [6], and there we gave a complete classification of those 3-manifolds that admit a contact circle satisfying a certain additional volume constraint.

In the present paper, whose methods are independent of those employed in [6], we show that every (closed, orientable) 3-manifold admits a contact circle.

1. Outline

We begin by recalling the precise definition of a contact circle.

Definition 1.1. A *contact circle* on a 3-manifold is a pair of contact forms (ω_1, ω_2) such that any non-trivial linear combination $\lambda_1\omega_1 + \lambda_2\omega_2$ with constant coefficients $(\lambda_1, \lambda_2) \neq (0,0)$ is again a contact form.

In other words, we call a pair of 1-forms (ω_1, ω_2) a contact circle if

$$(\lambda_1\omega_1 + \lambda_2\omega_2) \wedge (\lambda_1\, d\omega_1 + \lambda_2\, d\omega_2)$$

is a volume form for all $(\lambda_1, \lambda_2) \neq (0,0)$, and it clearly suffices to check this condition for (λ_1, λ_2) with $\lambda_1^2 + \lambda_2^2 = 1$, hence the name.

Received May 30, 1996. The first author was partially supported by a Research Fellowship at Queens' College, Cambridge, and the second author by the Isaac Newton Institute, Cambridge, and by the Forschungsinstitut für Mathematik, ETH Zürich.

Analogously one defines a *contact sphere* to be a triple of contact forms $(\omega_1, \omega_2, \omega_3)$ such that $\lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3$ is a contact form for $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$.

The main result of this paper is the following.

Theorem 1.2. *On every closed, orientable* 3*-manifold there are contact circles realizing any of the two orientations.*

It is well-known that every 3-manifold admits a contact form, and virtually any structure theorem for 3-manifolds can be used to give a proof of this fact. The first proof, due to Martinet [13], was based on Lickorish's surgery description of 3-manifolds. Later two very short proofs were found, one by Thurston and Winkelnkemper [17], who used Alexander's open book decomposition for 3-manifolds, and one by the second-named author [10], who gave a proof based on a branched cover description of 3-manifolds due to Hilden, Montesinos, and Thickstun.

It turns out, however, that none of these proofs can be adapted directly to yield a proof of Theorem 1.2. While our proof of the main theorem is also based on Lickorish's surgery description, we have to control the position of the surgery curves relative to the common kernel of ω_1 and ω_2, whereas Martinet used surgery curves transverse to a given contact structure (i.e., the plane field defined by a contact form). Moreover, in Martinet's proof one can perform one elementary surgery at a time, whereas here we have to control all framings simultaneously.

Our initial attempts to understand which 3-manifolds admit a contact circle went in two directions. On the one hand, we studied a more restricted class of contact circles, so-called *taut* contact circles, which are characterized by the property that $\lambda_1\omega_1 + \lambda_2\omega_2$ defines the same volume form for all $\lambda_1^2 + \lambda_2^2 = 1$. In [6] we developed a theory for this structure and proved the following existence theorem, which contrasts sharply with Theorem 1.2.

Theorem 1.3. *A closed, orientable* 3*-manifold M admits a taut contact circle if and only if M is a quotient of the Lie group G by a discrete subgroup acting by left-multiplication, where G is one of* $\mathrm{SU}(2)$, $\widetilde{\mathrm{SL}}_2$ *(the universal cover of* $\mathrm{PSL}_2\mathbb{R}$*), or* $\widetilde{\mathrm{E}}_2$ *(the universal cover of the group of orientation preserving isometries of the Euclidean plane).*

In a different direction, and motivated by the manifolds occuring in Theorem 1.3, we constructed explicit examples of contact circles on other geometric manifolds. It appears that such explicit constructions are possible on all but the hyperbolic geometries. Also, one can give an elementary connected sum construction for the contact circles on some of these manifolds.

These examples are of course subsumed in Theorem 1.2, but we include them in Section 5 of the present paper because they provide very simple global descriptions of contact circles. For instance, one can show that all these explicit examples consist of tight contact structures, whereas it is not clear at all whether the rather intricate construction used to prove Theorem 1.2 yields tight or overtwisted contact structures.

Call a contact circle (resp. sphere) *overtwisted* if at least one of the contact forms it contains is overtwisted (In the case of a closed manifold, this implies that all contact forms in the circle (resp. sphere) are overtwisted by Gray's stability theorem). While we do not know of an explicit example of an overtwisted contact circle on a closed manifold, it is possible to construct an overtwisted contact sphere on $\mathbb{R}^3$. This construction is contained in Section 6. We obtain an overtwisted contact structure on $\mathbb{R}^3$ by pulling back the standard (tight) contact structure under a suitable immersion $\mathbb{R}^3 \to \mathbb{R}^3$, and by the same map we can pull back the standard contact sphere on $\mathbb{R}^3$.

We believe that the explicit examples in Section 5 and the construction of an overtwisted contact structure on $\mathbb{R}^3$ in Section 6 are of interest in contact geometry proper, quite apart from their significance for the construction of contact circles.

Section 5 also contains examples of contact spheres. We do not say anything about the existence of contact spheres in general; however, there is evidence that there may well be non-trivial obstructions to their existence, which should prove interesting from the viewpoint of 3-manifold topology. We hope to address this issue in a later publication.

In Section 2 we state two modification lemmas and two extension lemmas for contact circles. The second extension lemma gives a criterion for when a contact circle given near the boundary of a solid torus can be extended to the inside. We then give a proof of Theorem 1.2 based on these lemmas.

In Section 3 we are concerned with the local geometry of contact circles and prove the modification lemmas; in Section 4, the extension lemmas.

Some of the results in the present paper were announced in [7]; there and in the introduction to [6] the reader can find more on the motivation to study contact circles. Section 5 of the present paper is largely identical with the previously circulated preprint "Contact circles and tight contact structures on geometric 3-manifolds."

2. Proof of the Main Theorem

In this section we state the modification lemmas and the extension lemmas for contact circles, and then proceed to give a proof of Theorem 1.2. The proofs of the modification lemmas will be deferred to Section 3 and the proofs of the extension lemmas to Section 4.

Let (ω_1, ω_2) be a contact circle on $\Sigma \times [-1, 1]$, where Σ is a compact, orientable, connected surface (with or without boundary). Let t denote the coordinate in $[-1, 1]$. We assume that the common kernel $\ker \omega_1 \cap \ker \omega_2$ is spanned by ∂_t. Then the 1-forms ω_1, ω_2 define a parallelization $(\omega_{1t}, \omega_{2t}) \stackrel{\text{def}}{=} (\omega_1, \omega_2)|T(\Sigma \times \{t\})$ of each slice $\Sigma \times \{t\}$ (In particular, Σ must be a torus or have non-empty boundary). This allows to measure (with sign) the rotation of the frame $(\omega_{1t}, \omega_{2t})$, with respect to the one induced on $\Sigma \times \{-1\}$, along any flow line of ∂_t.

The following two lemmas will be referred to as the modification lemmas.

Lemma 2.1. *Let (ω_1, ω_2) be a contact circle on $\Sigma \times [-1, 1]$ as described. Then, for any $\epsilon \in (0, 1)$, we can find a new contact circle (ω_1', ω_2') on $\Sigma \times [-1, 1]$ with the following properties:*

(i) $\ker \omega_1' \cap \ker \omega_2'$ *is spanned by* ∂_t.

(ii) (ω_1', ω_2') *coincides with* (ω_1, ω_2) *on* $\Sigma \times [-1, -\epsilon] \cup \Sigma \times [\epsilon, 1]$.

(iii) $(\omega_{1t}', \omega_{2t}')$ *makes precisely one more full twist by $\pm 2\pi$ than $(\omega_{1t}, \omega_{2t})$ along any segment $p \times [-1, 1]$, $p \in \Sigma$. The sign in $\pm 2\pi$ depends on the ambient orientation defined by ω_1.*

Lemma 2.2. *Suppose that on $\Sigma \times [-1, 1]$ we have a contact circle (ω_1, ω_2) which is smooth except along $\Sigma \times \{0\}$, where it is only continuous. Suppose further that the common kernel is spanned by ∂_t and that the orientations defined by $\omega_1 \wedge d\omega_1$ on $\Sigma \times [-1, 0]$ and on $\Sigma \times [0, 1]$ agree. Then there is an everywhere smooth contact circle which agrees with (ω_1, ω_2) near $\Sigma \times \{-1\}$ and near $\Sigma \times \{1\}$, and with common kernel still spanned by ∂_t.*

The following extension lemmas are homotopical in nature.

Lemma 2.3. *Suppose we have two homotopic parallelizations of Σ, and consider them as parallelizations of $\Sigma \times \{0\}$ and $\Sigma \times \{1\}$, respectively. There is a contact circle on $\Sigma \times [0, 1]$ which induces these parallelizations, whose contact forms define any given orientation, and whose common kernel is spanned by ∂_t.*

Lemma 2.4. *Let (ω_1, ω_2) be a contact circle defined near the bound-*

ary $T^2 = S^1 \times \partial D^2$ of a solid torus $S^1 \times D^2$, with common kernel transverse to T^2. If the parallelization induced by (ω_1, ω_2) on T^2 has odd rotation number around the meridian with respect to the Lie group framing, then the contact circle extends inside over the whole solid torus.

Notice that on an oriented 3-manifold a pair (ω_1, ω_2) of pointwise linearly independent 1-forms determines a parallelization unique up to homotopy. Then Lemma 2.4 implies that the obstruction to extending (ω_1, ω_2) as a contact circle is the same as the obstruction to extending its induced parallelization (for the parallelization defined along the meridian represents the trivial element of $\pi_1(\mathrm{SO}_3) \cong \mathbb{Z}_2$ precisely if the mentioned rotation number is odd). Since a contact form induces a natural orientation, any construction of a contact circle also yields a natural parallelization. Our construction is inspired by the spin structure construction of a parallelization from a surgery description of the 3-manifold.

Taking the lemmas above for granted, we now turn to the proof of Theorem 1.2. In fact, only Lemma 2.1 and Lemma 2.4 are used directly in this proof. Given a closed, orientable 3-manifold M, we can represent it according to Lickorish [12] as follows (cf. [14]).

Start with a solid torus $S^1 \times D^2$, standardly embedded in S^3. Denote a meridian $* \times \partial D^2$ by λ_∞ and a longitude $S^1 \times *$ (with $* \in \partial D^2$) by $\sigma_\infty = \mu_\infty$. (In using this notation we think of $\lambda_\infty, \mu_\infty$ as longitude and meridian of a complementary solid torus $S^3 - \mathrm{int}\,(S^1 \times D^2)$. In particular, the longitude μ_∞ is homologically trivial in this complementary solid torus.) A great circle γ in $S^1 \times D^2$ is, by definition, the graph of a map $S^1 \to \mathrm{int} D^2$. Remove from $S^1 \times D^2$ a finite family of disjoint open tubular neighbourhoods $\mathrm{int} N_i$ of great circles γ_i, $i = 1, ..., n$. Denote the resulting manifold with boundary by Ω. The N_i are solid tori, and there is a (homologically) well-defined meridian μ_i on ∂N_i which generates the kernel of $\pi_1(\partial N_i) \to \pi_1(N_i)$. There is a distinguished longitude λ_i on ∂N_i which is homologically trivial in $S^3 - \mathrm{int} N_i$. The λ_i and μ_i generate $\pi_1(\partial N_i)$.

Now choose surgery curves σ_i on ∂N_i homologous to $\mu_i \pm \lambda_i$, $i = 1, ..., n$. Finally, glue in solid tori $V_i \cong S^1 \times D^2$, $i = 1, ..., n, \infty$, such that σ_i becomes a meridian in V_i, to obtain a closed 3-manifold.

For any given M, the choices in the above construction can be made in such a way that the resulting manifold is diffeomorphic to M.

We may assume without loss of generality that the ∂N_i are transverse to the slices $\{\theta\} \times D^2$ for all $\theta \in S^1$. In other words, at any point $(\theta, x, y) \in S^1 \times D^2$ we find a vector tangent to ∂N_i of the form $\partial_\theta + a\partial_x + b\partial_y$.

Lemma 2.5. *There is a contact circle on Ω with common kernel tangent to $\partial\Omega$ and defining any given ambient orientation.*

Proof. Consider the contact circle (ω_1, ω_2) on $S^1 \times D^2$ defined by

$$\omega_1 = \cos\theta\, dx - \sin\theta\, dy,$$

$$\omega_2 = \sin\theta\, dx + \cos\theta\, dy.$$

This has common kernel spanned by ∂_θ. Since the contact circle condition is open in the C^1-topology, it is possible to perturb this to a contact circle (ω_1', ω_2') on $S^1 \times D^2$ with common kernel spanned by $\partial_\theta + a\partial_x + b\partial_y$ if a and b are C^1-small functions on $S^1 \times D^2$.

Now consider the contact circle $(\omega_{K,1}, \omega_{K,2})$ defined by

$$\omega_{K,1} = \cos(K\theta)\, dx - \sin(K\theta)\, dy,$$

$$\omega_{K,2} = \sin(K\theta)\, dx + \cos(K\theta)\, dy.$$

We have $\omega_{K,i} = \phi_K^* \omega_i$, where $\phi(\theta, x, y) = (K\theta, x, y)$. The perturbed contact circle $(\omega_{K,1}', \omega_{K,2}')$ defined by $\omega_{K,i}' = \phi_K^* \omega_i'$ has common kernel spanned by

$$K(\phi_{K*})^{-1}(\partial_\theta + a\partial_x + b\partial_y) = \partial_\theta + K(a \circ \phi)\partial_x + K(b \circ \phi)\partial_y.$$

This shows that the allowable perturbation of the common kernel increases as $|K|$ grows larger. Hence, for $|K|$ sufficiently large and for the appropriate sign of K, the contact circle $(\omega_{K,1}, \omega_{K,2})$ can be perturbed to a contact circle $(\omega_{K,1}', \omega_{K,2}')$ with the desired property.

Let Σ_∞ be the element in $H_2(\Omega, \partial\Omega)$ represented by $(* \times D^2) \cap \Omega$. By slight abuse of notation we shall sometimes identify Σ_∞ (and other elements of $H_2(\Omega, \partial\Omega)$) with a particular surface representing this class.

By the definition of λ_i there is an element Σ_i in $H_2(\Omega, \partial\Omega)$ that is represented by an annulus with boundary curves μ_∞ and λ_i, and with a certain number of discs removed where the N_j, $j \neq i$, cut this annulus.

Denote the linking number of the great circles γ_i and γ_j by l_{ij}. We shall see that only the value of l_{ij} modulo 2 is relevant to our problem, so we do not have to worry about a sign convention for this linking number.

For Σ a surface in Ω with boundary on $\partial\Omega$ and σ a curve in Ω we write $\#(\sigma,\Sigma)$ for their intersection number. Again we only count this modulo 2.

Now we are ready to construct the contact circle on Ω which has the right behaviour near $\partial\Omega$ so that an application of Lemma 2.4 yields a contact circle on M. Start with the contact circle $(\omega'_{K,1},\omega'_{K,2})$ on Ω with common kernel tangent to $\partial\Omega$, as constructed in Lemma 2.5, where we take K to be even and of the appropriate sign for the given ambient orientation. If this contact circle is now perturbed slightly to make the common kernel transverse to $\partial\Omega$, the induced parallelization on the boundary tori makes K turns (with respect to the Lie group framing) along the longitudes $\lambda_1,...\lambda_n,\mu_\infty$, and ± 1 turn along the meridians $\mu_1,...,\mu_n$. Hence it makes an odd number of turns along the surgery curves $\sigma_1,...,\sigma_n$, and an even number of turns along the surgery curve $\sigma_\infty=\mu_\infty$.

We now want to apply Lemma 2.1 to change the parity of the number of turns along σ_∞, while keeping the parity along $\sigma_1,...,\sigma_n$. We perform this construction with the initial contact circle with common kernel tangent to $\partial\Omega$, because any change in the number of full twists (in the sense of Lemma 2.1) of this initial contact circle (along a given curve on $\partial\Omega$) will produce the same change in the number of turns of the slightly perturbed contact circle with common kernel transverse to $\partial\Omega$ (along the same curve). In order to make this change in the number of full twists, we consider an element $\Sigma\in H_2(\Omega,\partial\Omega)$ of the form

$$\Sigma=\sum_{i=1}^{n}a_i\Sigma_i+a_\infty\Sigma_\infty.$$

Lemma 2.6. *For a_∞ sufficiently large, Σ can be represented by a union of n properly embedded surfaces S_i, which need not be pairwise disjoint, each transverse to the common kernel of $\omega'_{K,1}$ and $\omega'_{K,2}$.*

Assuming this lemma, we can introduce an additional twist into the contact circle when we pass the surface S_1, then the same for the surface S_2, and so on. Here we identify S_i with $S_i\times 0$ and think of $S_i\times[-1,1]$ as a tubular neighbourhood of S_i in Ω.

Now we have the following intersection numbers (mod 2):

$$\begin{aligned}\#(\sigma_\infty,\Sigma_\infty) &= 1,\\ \#(\sigma_\infty,\Sigma_i) &= 0, \quad i=1,...,n.\end{aligned}$$

Since we want to change the parity of the number of twists along σ_∞, we require

$$a_\infty \equiv 1 \mod 2.$$

Secondly, we have for $i = 1, ..., n$:

$$\begin{array}{lcl} \#(\sigma_i, \Sigma_\infty) & = & 1, \\ \#(\sigma_i, \Sigma_i) & = & 1, \\ \#(\sigma_i, \Sigma_j) & = & l_{ij} \quad \text{for } i \neq j, \end{array}$$

again mod 2. Since we want to keep the parity along σ_i, we stipulate

$$a_i + \sum_{j \neq i} l_{ij} a_j + a_\infty \equiv 0 \mod 2, \quad i = 1, ..., n.$$

The proof of Theorem 1.2 is then completed by the following simple algebraic lemma.

Lemma 2.7. *The linear system of equations* (*over* $\mathbb{Z}_2$)

$$a_i + \sum_{j \neq i} l_{ij} a_j \equiv 1 \mod 2,$$

where $l_{ij} = l_{ji}$, *always has a solution.*

Indeed, we define a class Σ by a solution to the system of equations in Lemma 2.7. For a_∞ sufficiently large (and odd) we may assume by Lemma 2.6 that Σ is represented by a union $S_1 \cup \ldots \cup S_n$ of properly embedded surfaces transverse to the common kernel of $\omega'_{K,1}$ and $\omega'_{K,2}$. We then perform the twisting along S_1 as in Lemma 2.1, then along S_2, and continue up to S_n. Then we slightly perturb the resulting contact circle to make the common kernel transverse to $\partial\Omega$, and we get a contact circle on Ω which induces a parallelization of $\partial\Omega$ making an odd number of turns along each surgery curve (with respect to the Lie group framing of the respective torus component of $\partial\Omega$). Finally, Lemma 2.4 applies to yield a contact circle on M. This concludes the proof of Theorem 1.2.

Proof of Lemma 2.6. The problem can be reduced to the following. Let Ω be a solid torus $S^1 \times D^2$ with a concentric solid torus N_1 removed. Let Σ_1 and Σ_∞ be as before. We claim that for a_∞ sufficiently large the homology class

$$a_1\Sigma_1 + a_\infty\Sigma_\infty \in H_2(\Omega, \partial\Omega)$$

can be represented by a properly embedded surface transverse to a given vector field $\partial_\theta + a\partial_x + b\partial_y$, in other words, that we can make the angle between ∂_θ and the embedded surface as close to $\pi/2$ as we wish.

For us it is enough to consider the cases $a_1 = 0$ and $a_1 = 1$, although the lemma also holds true for integral (rather than $\mathbb{Z}_2$) homology classes. Clearly the first case is trivial, hence consider

$$\Sigma = \Sigma_1 + a_\infty \Sigma_\infty.$$

Now there is an obvious way to replace this surface by a homologically equivalent one which lifts to a helicoid in the universal cover (see Fig. 1, where the boundary curves of Σ on the outer boundary of Ω are shown) and which for a_∞ large has the desired property.

For several disjoint solid tori N_i inside $S^1 \times D^2$, we get helicoid-like surfaces (with discs removed), each embedded and transverse to the vector field $\partial_\theta + a\partial_x + b\partial_y$, although perhaps not pairwise disjoint. These helicoids represent the classes $\Sigma_i + a_i'\Sigma_\infty$, where the a_i' are all sufficiently large for the transversality to be achieved and their sum equal to a_∞. Each of these surfaces is topologically a disc with some interior open discs removed.

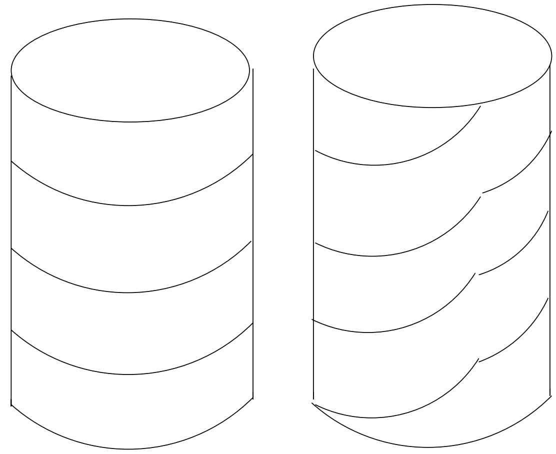

FIGURE 1. Helicoid in Ω.

Proof of Lemma 2.7. Set $l_{ii} = 1$ and let L be the $(n \times n)$-matrix (l_{ij}) and $\mathbf{a}$ the column vector with entries $a_1, \dots, a_n$. Let $\mathbf{1}$ be the column vector with all n entries equal to 1. Then the linear system of equations can be written as $L\mathbf{a} \equiv \mathbf{1} \bmod 2$. To show that this system has a solution for any choice of L (with $l_{ij} = l_{ji}$ and $l_{ii} = 1$) it is sufficient to show that any linear relation satisfied by the rows of L is also satisfied by the rows (i.e., entries) of $\mathbf{1}$ (modulo 2). This reduces to showing that the sum of k rows of L can only be the zero row if k is even. By exchanging the columns it suffices to show this for the sum of the first k rows. If this sum is the zero row, then in particular $\sum_{i,j=1}^{k} l_{ij} \equiv 0 \bmod 2$, hence $k \equiv \sum_{i<j} 2l_{ij} \equiv 0 \bmod 2$.

3. Local study of contact circles

Our aim now is to lay the foundation for the proof of the modification lemmas and the extension lemmas. The starting point is a local construction which is flexible enough to allow certain glueing operations. In order to have some control over the contact circle property we impose a weak condition, namely, we specify a simple foliation by curves and look for those contact circles which have these as integral curves of the common kernel $\ker\omega_1 \cap \ker\omega_2$.

We shall further request that $\omega_1 \wedge \omega_2$ define in some domain an invariant transverse measure for this foliation. This we can often achieve just by passing from (ω_1, ω_2) to $(h\omega_1, h\omega_2)$, where h is a positive function, and this will simplify computations considerably.

Let a coordinate system (u, v, w) be given and suppose that the common kernel $\ker\omega_1 \cap \ker\omega_2$ is spanned by the coordinate vector field ∂_w. Thus there are four functions x_1, x_2, x_3, x_4 of (u, v, w) such that

$$\omega_1 = x_1\, du + x_2\, dv,$$

$$\omega_2 = x_3\, du + x_4\, dv.$$

The pair (ω_1, ω_2) is a contact circle if and only if the symmetric part of the matrix

$$\begin{pmatrix} \omega_1 \wedge d\omega_1 & \omega_1 \wedge d\omega_2 \\ \omega_2 \wedge d\omega_1 & \omega_2 \wedge d\omega_2 \end{pmatrix}$$

is definite. If we write $\omega_i \wedge d\omega_j = a_{ij}\, du \wedge dv \wedge dw$, then we compute

$$4a_{11}a_{22} - (a_{12} + a_{21})^2 = Q(\mathbf{x}_w),$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mathbf{x}_w = d\mathbf{x}(\partial_w)$, and Q is the following field of quadratic forms:

$$Q = 4(x_1x_4 - x_2x_3)(dx_1dx_4 - dx_2dx_3) - (d(x_1x_4 - x_2x_3))^2.$$

Thus the contact circle property is equivalent to $Q(\mathbf{x}_w) > 0$.

It is straightforward to compute

$$\omega_1 \wedge \omega_2 = (x_1 x_4 - x_2 x_3)\, du \wedge dv.$$

By passing to the pair $(h\omega_1, h\omega_2)$ for suitable $h > 0$, we may assume $\omega_1 \wedge \omega_2 = du \wedge dv$ and this forces $x_1 x_4 - x_2 x_3 \equiv 1$. Now the condition $Q(\mathbf{x}_w) > 0$ simplifies to $Q_0(\mathbf{x}_w) > 0$, where

$$Q_0 = dx_1 dx_4 - dx_2 dx_3.$$

Since $Q(\mathbf{x}_w) > 0$ is a condition along each orbit of ∂_w, it actually suffices to require that $x_1 x_4 - x_2 x_3$ be constant along such curves to ensure that $Q(\mathbf{x}_w) = f Q_0(\mathbf{x}_w)$ with some positive function f, but for the sake of simplicity we shall assume $x_1 x_4 - x_2 x_3 \equiv 1$.

It is now natural to change to new coefficient functions y_1, y_2, y_3, y_4 given by

$$\omega_1 = (y_1 + y_3)\, du + (y_4 - y_2)\, dv,$$

$$\omega_2 = (y_2 + y_4)\, du + (y_1 - y_3)\, dv,$$

because in terms of these we have

$$\begin{aligned} \omega_1 \wedge \omega_2 &= (y_1^2 + y_2^2 - y_3^2 - y_4^2)\, du \wedge dv, \\ Q_0 &= dy_1^2 + dy_2^2 - dy_3^2 - dy_4^2. \end{aligned}$$

Let S denote the quadric in $\mathbb{R}^4$ defined by the equation $y_1^2 + y_2^2 - y_3^2 - y_4^2 = 1$ and let $j : S \to \mathbb{R}^4$ be the inclusion map. S is diffeomorphic to $S^1 \times \mathbb{R}^2$. Then

$$(S, g) = (S, j^*(dy_1^2 + dy_2^2 - dy_3^2 - dy_4^2))$$

is a Lorentzian manifold with signature $+--$ (Fig. 2).

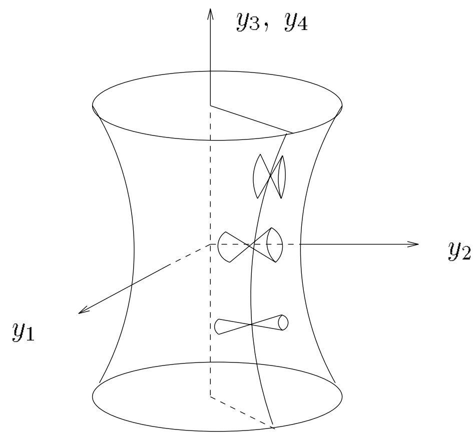

FIGURE 2. The Lorentzian manifold S.

Call a tangent vector *time-like* for g if the value of g on this vector is positive. Call a curve on S time-like if g induces a positive quadratic form on its tangent lines, i.e., if the non-zero tangent vectors to the curve are time-like.

This discussion shows the following, which we formulate as a separate lemma for future reference.

Lemma 3.1. *A pair* (ω_1, ω_2) *with* $\omega_1 \wedge \omega_2 = du \wedge dv$ *and common kernel spanned by* ∂_w *is a contact circle, if and only if for each* (u_0, v_0) *the curve in* S *given by*

$$\gamma(w) = (y_1, y_2, y_3, y_4)_{(u_0, v_0, w)}$$

is time-like for g.

Now introduce polar coordinates $(y_1, y_2) = (r \cos\varphi, r \sin\varphi)$. Call a curve in S *horizontal* if y_3 and y_4 are constant on it. Notice that any horizontal circle on S given by $r =$ constant is a time-like curve. We denote by ∂_φ the velocity field of the parametric curves

$$(y_1, y_2, y_3, y_4) = (r_0 \cdot \cos\varphi, r_0 \cdot \sin\varphi, c_3, c_4),$$

for all triples of constants r_0, c_3, c_4 such that $r_0^2 = c_3^2 + c_4^2 + 1$ and $r_0 > 0$. This defines a time-like vector field on S.

A contact circle (ω_1, ω_2) defines a 3-dimensional orientation (or ambient orientation) where the forms $\omega \wedge d\omega$ are positive for any $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$ with λ_1 and λ_2 constants.

In the geometry (S, g) the time-like vectors form two disjoint open solid cones, which correspond to the two possible ambient orientations defined by the contact circle.

For the case of $y_3 = y_4 \equiv 0$ and $r \equiv 1$, we get

$$\omega_1 \wedge d\omega_1 = \frac{\partial \varphi}{\partial w} du \wedge dv \wedge dw.$$

Thus Lemma 3.1 has the following refinement.

Lemma 3.2. *Under the hypotheses of Lemma* 3.1 *the contact circle* (ω_1, ω_2) *defines the orientation of* $du \wedge dv \wedge dw$*, if and only if the time-like paths it induces on* S *have their velocity vectors in the component of the time-like cones containing the vectors* ∂_φ*.*

The local geometric study we are developing also provides a simple condition for the pair (ω_1, ω_2) to be a *taut* contact circle. A contact circle (ω_1, ω_2) is taut if and only if the following identities hold:

$$\begin{aligned} \omega_1 \wedge d\omega_1 - \omega_2 \wedge d\omega_2 &\equiv 0, \\ \omega_1 \wedge d\omega_2 + \omega_2 \wedge d\omega_1 &\equiv 0. \end{aligned}$$

In our case we compute

$$\begin{aligned} &\omega_1 \wedge d\omega_1 - \omega_2 \wedge d\omega_2 \\ &\qquad = 2\langle y_4 dy_1 - y_1 dy_4 + y_3 dy_2 - y_2 dy_3, \partial_w \rangle du \wedge dv \wedge dw, \\ &\omega_1 \wedge d\omega_2 + \omega_2 \wedge d\omega_1 \\ &\qquad = 2\langle y_4 dy_2 - y_2 dy_4 + y_1 dy_3 - y_3 dy_1, \partial_w \rangle du \wedge dv \wedge dw. \end{aligned}$$

Multiplication of both 1-forms by the same positive function preserves the taut contact circle property, so we assume as above that we have $\omega_1 \wedge \omega_2 \equiv du \wedge dv$. Then the contact circle is taut if and only if the vector $\mathbf{x}_w$ satisfies the following Pfaffian system:

$$\begin{aligned} A_1 &\equiv y_1 dy_1 + y_2 dy_2 - y_3 dy_3 - y_4 dy_4 = 0, \\ A_2 &\equiv y_4 dy_1 - y_1 dy_4 + y_3 dy_2 - y_2 dy_3 = 0, \\ A_3 &\equiv y_4 dy_2 - y_2 dy_4 + y_1 dy_3 - y_3 dy_1 = 0, \end{aligned}$$

where the first equation demands that the vector be tangent to S. It is straightforward to check that $A_1 \wedge A_2 \wedge A_3$ equals

$$-(y_1^2 + y_2^2 - y_3^2 - y_4^2)\, i(Y)(dy_1 \wedge dy_2 \wedge dy_3 \wedge dy_4),$$

where

$$Y = y_1\partial_{y_2} - y_2\partial_{y_1} + y_3\partial_{y_4} - y_4\partial_{y_3},$$

and $i(Z)\eta$ denotes the interior product of a vector Z with a differential form η. So the Pfaffian system has rank 3 outside the cone $\{Q_0 = 0\}$ and defines the line field spanned by the vector field Y. This is the standard Hopf vector field in $\mathbb{R}^4$. Along S it is tangent to S, and we call its orbits the *Hopf fibres* on S.

Lemma 3.3. *Under the hypotheses of Lemma* 3.1 *the pair* (ω_1, ω_2) *is a taut contact circle, if and only if the paths it induces on* S *go along Hopf fibres on* S *and have non-zero speed.*

The discussion above translates to a more general setup. Let (ω_1, ω_2) be a pair of pointwise linearly independent 1-forms and assume X is any nowhere zero vector field spanning their common kernel. The coordinate chart (u, v, w) can obviously be substituted by any flow box for X of the form $\Sigma \times I$, where I is an interval and Σ is any connected, parallelizable surface. We restrict ourselves to the case of compact Σ, so that it must be a torus or have non-empty boundary. Moreover, in Lemma 2.4 only the case $\Sigma = T^2$ is used, and in the proof of Theorem 1.2 we use Lemma 2.1 only for Σ equal to a disc, possibly with some open discs removed. Thus, in order to simplify our argument, we henceforth disregard other possibilities and think of Σ as a torus or a compact planar domain with smooth boundary. The interval I is going to be compact in all cases considered.

To each orbit $\mathcal{O}$ of X in the flow box we associate a copy $\mathbf{V}_{\mathcal{O}}$ of our space $\mathbb{R}^4$ with coordinates y_1, y_2, y_3, y_4. To be more precise, we let $\mathbf{V}_{\mathcal{O}}$ represent the space of pairs $\mathbf{v} = (\widetilde{\omega}_1, \widetilde{\omega}_2)$ of (not necessarily linearly independent) 1-forms on the 3-manifold along $\mathcal{O}$ whith $\widetilde{\omega}_1(X) = \widetilde{\omega}_2(X) = 0$ and invariant under the flow of X. There is a unique symmetric, bilinear form $\langle\cdot,\cdot\rangle$ on this space such that $Q_0(\mathbf{v}) = (1/2)\langle\mathbf{v},\mathbf{v}\rangle$ for all $\mathbf{v}$. Written in terms of the exterior product, this bilinear form is given by

$$\langle(\widetilde{\omega}_1, \widetilde{\omega}_2), (\widetilde{\eta}_1, \widetilde{\eta}_2)\rangle = \widetilde{\omega}_1 \wedge \widetilde{\eta}_2 + \widetilde{\eta}_1 \wedge \widetilde{\omega}_2,$$

considered as a scalar multiple of $du \wedge dv$. This defines a quadric $S_{\mathcal{O}} \subset \mathbf{V}_{\mathcal{O}}$ for each orbit $\mathcal{O}$.

We arrive at the same condition $Q(\mathbf{x}_w) > 0$ for the pair (ω_1, ω_2) to be a contact circle, if in the definition of the components x_i, a_{ij}, and y_i we replace the basis du, dv by any other basis α, β for the annihilator of

X with the property that α and β are invariant under the flow of X. A quick way to see this is to define the coefficients a_{ij} not by the identities

$$\omega_i \wedge d\omega_j = a_{ij}\alpha \wedge \beta \wedge dw,$$

but by the following equivalent identities

$$\omega_i \wedge L_X\omega_j = -a_{ij}\alpha \wedge \beta.$$

So the quadric $S_{\mathcal{O}}$ is defined by giving any 2-form on the 3-manifold along $\mathcal{O}$ invariant under the flow of X and annihilated by $i(X)$. Given a pair (ω_1, ω_2) with common kernel spanned by X we may assume, modulo multiplication by a function, that $\omega_1 \wedge \omega_2$ equals that 2-form identically on the flow box, and now to each orbit $\mathcal{O}$ of X there is an associated path γ in $S_{\mathcal{O}}$. We have shown that the contact circle properties of the given pair correspond to geometric properties of these paths within their respective quadrics.

Notice that (du, dv) is the point whose y_i-coordinates are $(1, 0, 0, 0)$, and $(-dv, du)$ is the point with y_i-coordinates $(0, 1, 0, 0)$. Therefore the coordinate functions y_1 and y_2 can be defined by the following formulas

$$\begin{aligned} y_1 &= \frac{1}{2}\langle (du, dv), \mathbf{v}\rangle, \\ y_2 &= \frac{1}{2}\langle (-dv, du), \mathbf{v}\rangle. \end{aligned}$$

The vectors $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$ span a plane $\mathcal{P}$ which is positive definite for the quadratic form Q_0, the Q_0-orthogonal complement $\mathcal{P}^\perp$ being the span of $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. The splitting $\mathbf{V}_{\mathcal{O}} = \mathcal{P} \oplus \mathcal{P}^\perp$ then provides a definition of the polar coordinate r, because r^2 is the projection onto $\mathcal{P}$ followed by the quadratic form Q_0. The conclusion is that the quadric $S_{\mathcal{O}}$ and the new Lorentzian metric $r^{-2}g$ on $S_{\mathcal{O}}$ are determined by a flow-invariant Riemannian metric on the transversals of the flow of X; a choice of a flow-invariant transverse orthonormal frame determines the coordinates y_i. As we shall see, it is that metric $r^{-2}g$ which is best suited to our discussion. After having clarified the interpretation of $S_{\mathcal{O}}$ we revert to our original notation and suppress the suffix $\mathcal{O}$.

For each value of φ, define a half 3-plane $H_\varphi \subset \mathbb{R}^4$ by

$$(y_1, y_2) = r \cdot (\cos\varphi, \sin\varphi), \quad r > 0, (y_3, y_4) \text{ arbitrary}.$$

The intersection $S_\varphi \stackrel{\mathrm{def}}{=} H_\varphi \cap S$ is a connected component of a 2-sheeted hyperboloid (Fig. 3). The vector field ∂_φ is g-orthogonal to these surfaces S_φ. Observe that S_φ with the metric induced from $-g$ is a hyperbolic plane; indeed, this is the standard hyperboloid model of H^2.

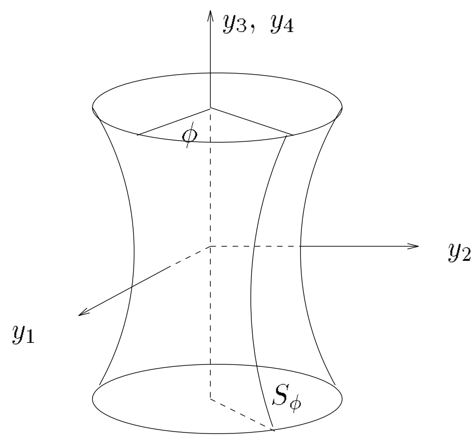

FIGURE 3. The hyperboloid S_φ.

We have $g(\partial_\varphi) = r^2$, and $r \geq 1$ everywhere on S. So we can divide g by r^2, and ∂_φ becomes a time-like vector field of constant length 1 in $(S, r^{-2}g)$, orthogonal to the surfaces S_φ.

Let S^2_+ be the hemisphere

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ x > 0\},$$

and let g_1 be the metric (with Gaussian curvature 1) induced on S^2_+ from the Euclidean metric of $\mathbb{R}^3$. It turns out that with the metric induced from $-r^{-2}g$ each surface S_φ is isometric to (S^2_+, g_1). The isometry is given by the formulas

$$\begin{aligned}(x, y, z) &= \frac{1}{r}(1, y_3, y_4),\\ (r, y_3, y_4) &= \frac{1}{x}(1, y, z).\end{aligned}$$

This, together with the orthogonality of ∂_φ and the foliation $\{S_\varphi\}$, gives us an obvious isometry between $(S, r^{-2}g)$ and $(S^1 \times S^2_+, d\varphi^2 - g_1)$. Giving a curve $\gamma(t) \subset S$ is equivalent to giving an S^1-valued function $\varphi(t)$ and a path $\gamma_1(t) \subset S^2_+$. The curve γ is time-like for g if and only if it is time-like for $r^{-2}g$, which in turn is equivalent to

$$\varphi'(t)^2 - g_1(\gamma_1'(t)) > 0 \text{ for all } t.$$

In particular, $\varphi'(t)$ is never zero, and by a change of parameter we may assume $|\varphi'(t)| \equiv 1$. In other words, a time-like curve can always be parametrized using $\pm\varphi$ (regarded as element in the universal cover $\mathbb{R}$ of S^1) as parameter. Now the condition for a curve $\gamma \subset S$ with such a parametrization to be time-like is

$$g_1(\gamma_1'(\varphi)) < 1 \text{ for all } \varphi.$$

Given a path $\gamma_1(\varphi) \subset S^2_+$ whose speed is less than 1 at every point (here φ ranges over an interval of any length), it induces two time-like paths on S whose respective descriptions on $S^1 \times S^2_+$ are

$$(\varphi, \gamma_1(\varphi)) \text{ and } (-\varphi, \gamma_1(\varphi)).$$

These two paths are symmetric about the hyperplane $\{y_2 = 0\}$.

For our purposes the most important properties of the Riemannian manifold (S^2_+, g_1) are that any two points can be joined by a *unique* geodesic segment and that the pole $(1, 0, 0)$ can be joined to every point by a geodesic segment whose length is strictly less than $\pi/2$.

Proposition 3.4. *Let Σ be a torus or a compact planar domain. Let (ω_1, ω_2) and (η_1, η_2) be two parallelizations of Σ satisfying the following conditions:*

(i) *They define the same orientation of Σ.*

(ii) *At each point of Σ, the 2-form $\omega_1 \wedge \eta_2 + \eta_1 \wedge \omega_2$ is a non-positive multiple of $\omega_1 \wedge \omega_2$.*

Consider $\Sigma \times [0, 1]$ and let t be the coordinate for the interval $[0, 1]$. Then each of the two ambient orientations of $\Sigma \times [0, 1]$ is realized by a contact circle with common kernel spanned by ∂_t and inducing the parallelization (ω_1, ω_2) on $\Sigma \times \{0\}$ and (η_1, η_2) on $\Sigma \times \{1\}$, respectively.

Proof. After multiplying the forms ω_1, ω_2 by the same positive function, we may assume

(i') $\omega_1 \wedge \omega_2 \equiv \eta_1 \wedge \eta_2$.

Now there is a unique 2-form on $\Sigma \times [0, 1]$ invariant under the flow of $X = \partial_t$, annihilated by the interior product with X and inducing $\omega_1 \wedge \omega_2$

on $\Sigma\times\{0\}$ (resp. $\eta_1\wedge\eta_2$ on $\Sigma\times\{1\}$). We can use this 2-form to define a quadric $S_{\mathcal{O}}$ along each orbit $\mathcal{O}$ of X.

The parallelization (ω_1,ω_2) extends in a unique way to a basis (α,β), invariant under the flow of X, of the annihilator of X. This basis provides each quadric $S_{\mathcal{O}}$ with coordinates y_1,y_2,y_3,y_4 and consequently with a Lorentzian metric $r^{-2}g$. We identify ω_1 and ω_2 with 1-forms along $\Sigma\times\{0\}$ annihilating X, and likewise for η_1 and η_2 along $\Sigma\times\{1\}$. Then, in the y_i coordinates just defined, the value $(\omega_1,\omega_2)_p$ of (ω_1,ω_2) at any $p\in\Sigma$ is the point $(1,0,0,0)$ on the corresponding quadric $S_{\mathcal{O}}$, while, because of conditions (i') and (ii), the value $(\eta_1,\eta_2)_p$ is a point on $S_{\mathcal{O}}$ with $y_1\leq 0$. In fact the y_1-coordinate of $(\eta_1,\eta_2)_p$ is given by

$$y_1=\frac{1}{2}\langle(\eta_1,\eta_2)_p,(1,0,0,0)\rangle=\frac{1}{2}\langle(\eta_1,\eta_2)_p,(\omega_1,\omega_2)_p\rangle,$$

which is non-positive by (ii). In terms of the polar coordinates r,φ, the pair $(\eta_1,\eta_2)_p$ corresponds to a point on $S_{\mathcal{O}}$ with $\varphi\in[\pi/2,3\pi/2]$.

An equivalent formulation of the problem is now the following: Given a smooth map $f:\Sigma\to S$ whose image lies entirely in the region given by $\varphi\in[\pi/2,3\pi/2]$, construct a family of time-like paths $\gamma^p(t)$ on $(S,r^{-2}g)$, depending smoothly on $(p,t)\in\Sigma\times[0,1]$, each with initial point $(1,0,0,0)$ and endpoint $f(p)$, running in the direction of increasing φ; to obtain the opposite ambient orientation construct a similar family running in the direction of decreasing φ.

In terms of the isometry between $(S,r^{-2}g)$ and $(S^1\times S^2_+,d\varphi^2-g_1)$ we have $f(p)\equiv(\varphi(p),f_1(p))$ with smooth maps

$$\begin{array}{rccl}\varphi: & \Sigma & \longrightarrow & [\pi/2,3\pi/2]\subset S^1,\\ f_1: & \Sigma & \longrightarrow & S^2_+,\end{array}$$

and the desired families can be defined by

$$\gamma^p(t)=(\pm t\varphi(p),\gamma_1^p(t)),\ \ 0\leq t\leq 1,$$

where $\gamma_1^p(t)$ is the geodesic segment in S^2_+ with initial point $(1,0,0)$ and endpoint $f_1(p)$, parametrized proportionally to arclength. Since the total length of each such segment is strictly less than $\pi/2$ and t goes from 0 to 1, $\gamma_1^p(t)$ has speed less than $\pi/2$, hence less than the angular speed $\varphi(p)$ of γ^p. This implies that the paths γ^p are time-like. The smooth dependence of these paths on p is a direct consequence of the geometry of the hemisphere S^2_+.

Proof of Lemma 2.1. We start with the same constructions as in the preceding proof, using the contact forms at $t = -1$ as reference. Thus the contact circle becomes a family of time-like paths $\gamma^p(t)$ on S, depending smoothly on $(p,t) \in \Sigma \times [-1,1]$, which we describe in terms of the model $S^1 \times S^2_+$ by

$$\gamma^p(t) = (\varphi^p(t), \gamma_1^p(t)).$$

The derivative $\varphi^{p\prime}(t)$ is never zero, and we shall assume it is positive (the argument being analogous in the other case). Each path γ^p has $(1,0,0,0)$ as initial point, which translates to the conditions

$$\varphi^p(-1) = 0 \ \text{ and } \ \gamma_1^p(-1) = (1,0,0) \ \text{ for all } \ p \in \Sigma.$$

The time-like property becomes $\varphi^{p\prime}(t) > \|\gamma_1^{p\prime}(t)\|$, where the norm is taken with respect to the metric g_1 on the hemisphere.

If $\widetilde{\varphi}^p(t)$ is any smooth function on $\Sigma \times [-1,1]$ with $\widetilde{\varphi}^{p\prime}(t) \geq \varphi^{p\prime}(t)$ for all p and t, then the paths given by

$$(\widetilde{\varphi}^p(t), \gamma_1^p(t))$$

are time-like and they define a new contact circle which induces the same ambient orientation as (ω_1, ω_2).

It is clear that there are functions $\widetilde{\varphi}^p(t)$ (depending smoothly on p and t) which satisfy that inequality and are identical to $\varphi^p(t)$ for t near -1 and to $\varphi^p(t) + 2\pi$ for t near 1. They define contact circles (ω_1', ω_2') on $\Sigma \times [-1,1]$ which agree with (ω_1, ω_2) near the slices $t = -1$ and $t = 1$.

It remains to check that adding $+2\pi$ to the total increase of the polar coordinate φ amounts to adding -2π to the total twisting of the frame $(\omega_{1t}', \omega_{2t}')$ induced on the slices $\Sigma \times \{t\}$ (with both signs reversed in case of the opposite ambient orientation). We have

$$\omega_{1t}' = (y_1 + y_3)\alpha_{-1} + (y_4 - y_2)\beta_{-1},$$

where α_{-1}, β_{-1} is the frame induced by (ω_1, ω_2) on $\Sigma \times \{-1\}$. Write

$$\omega_{1t}' = \rho \cdot (\cos\overline{\varphi}\,\alpha_{-1} + \sin\overline{\varphi}\,\beta_{-1})$$

with ρ positive. Observe that the choice of ambient orientation given by $\varphi^{p\prime}(t) > 0$ corresponds to $\overline{\varphi}'(t) < 0$. Both the total increase of φ and the total decrease of $\overline{\varphi}$ are unchanged by homotopies of paths on S with endpoints fixed. Conversely two paths on S with the same endpoints and the same total increase of φ are homotopic with endpoints fixed because S is topologically $S^1 \times \mathbb{R}^2$, the angle function φ being the projection onto the first factor. In particular, for each $p \in \Sigma$ the corresponding path defined by (ω_1', ω_2') is homotopic to the (continuous) path which starts at $(1,0,0,0)$, goes along the horizontal circle on S defined by $y_3 = y_4 = 0$ in the direction of increasing φ, and then goes along the path corresponding to (ω_1, ω_2) at the same point p. But along that horizontal circle we have

$$\omega_{1t}' = y_1\alpha_{-1} - y_2\beta_{-1} = \cos\varphi\,\alpha_{-1} - \sin\varphi\,\beta_{-1},$$

and so $\overline{\varphi} = -\varphi$ on the circle. We conclude that, for each point on Σ, the total decrease of $\overline{\varphi}$ along $(\omega_{1t}', \omega_{2t}')$ is -2π plus the total decrease of $\overline{\varphi}$ along $(\omega_{1t}, \omega_{2t})$.

If (ω_1, ω_2) defines the other orientation of $\Sigma \times [-1,1]$, then the paths go in the direction of decreasing φ, and the total increase of $\overline{\varphi}$ along $(\omega_{1t}', \omega_{2t}')$ is now 2π plus the total increase of $\overline{\varphi}$ along $(\omega_{1t}, \omega_{2t})$.

Proof of Lemma 2.2. Again start with the same constructions as in the proof of Proposition 3.4, but take the forms at $t = 0$ as reference. We have a description of (ω_1, ω_2) as a family of paths on S, all passing through the point $(1,0,0,0)$ for $t = 0$. In terms of the model $S^1 \times S^2_+$, this family of paths looks like

$$(\varphi^p(t), \gamma_1^p(t)),$$

where both $\varphi^p(t)$ and $\gamma_1^p(t)$ are smooth on the domains $\Sigma \times [-1,0]$ and $\Sigma \times [0,1]$ and continuous everywhere. The orientation hypothesis means that the derivative $\varphi^{p\prime}(t)$ is either positive or negative on both domains. Assume, say, that it is positive. We then have

$$\|\gamma_1^{p\prime}(t)\| < \varphi^{p\prime}(t)$$

on $-1 \leq t \leq 0$ and on $0 \leq t \leq 1$, with the norm induced by the metric g_1 on the hemisphere. We want to keep this differential inequality satisfied when we replace $(\varphi^p(t), \gamma_1^p(t))$ by an everywhere smooth map $(\widetilde{\varphi}^p(t), \widetilde{\gamma}_1^p(t))$ which coincides with $(\varphi^p(t), \gamma_1^p(t))$ outside a neighborhood of the slice $t = 0$.

We first construct $\widetilde{\gamma}_1^p(t)$. Since $\Sigma \times [-1,1]$ is compact, there exists a constant $\epsilon \in (0, 1/2)$ such that

$$(1 + 2\epsilon)\|\gamma_1^{p\prime}(t)\| < \varphi^{p\prime}(t)$$

on the same two domains as above. For this ϵ, let $a_\epsilon(t)$ be a smooth function on $t \in [0,1]$ such that (Fig. 4)

$$\begin{aligned}
&0 \leq a_\epsilon(t) \leq 1 + 2\epsilon \ \text{ for } \ t \in [0,1],\\
&a_\epsilon(t) = 0 \ \text{ for } \ t \in [0, \epsilon^2],\\
&a_\epsilon(t) = 1 \ \text{ for } \ t \geq \epsilon,\\
&\int_0^\epsilon a_\epsilon(t)\, dt = \epsilon.
\end{aligned}$$

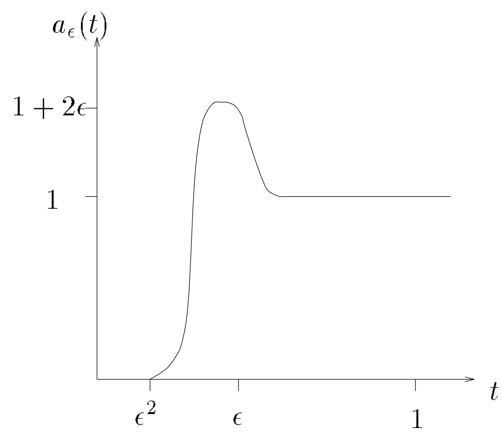

FIGURE 4. The function $a_\epsilon(t)$.

The function

$$\sigma_\epsilon(t) \stackrel{\text{def}}{=} \int_0^t a_\epsilon(|s|)\, ds, \quad t \in [-1,1],$$

is identically zero on $[-\epsilon^2, \epsilon^2]$ and satisfies $\sigma_\epsilon(t) = t$ for $|t| \geq \epsilon$. Moreover, it is non-decreasing and thus takes $[-1,1]$ to itself. Then the following family of paths on S^2_+,

$$\widetilde{\gamma}_1^p(t) \stackrel{\text{def}}{=} \gamma_1^p(\sigma_\epsilon(t)),$$

is constant equal to $(1,0,0)$ for $-\epsilon^2 \leq t \leq \epsilon^2$, which makes it everywhere smooth, coincides with $\gamma_1^p(t)$ for $|t| \geq \epsilon$, and satisfies

$$\|\widetilde{\gamma}_1^{p\prime}(t)\| = a_\epsilon(t)\|\gamma_1^{p\prime}(t)\| \leq (1+2\epsilon)\|\gamma_1^{p\prime}(t)\| < \varphi^{p\prime}(t) \ \text{ for all } \ t \in [0,1].$$

The function $\widetilde{\varphi}^p(t)$ has to be smooth and with everywhere positive derivative with respect to t. There exists a smooth function $b^p(t)$ on

$\Sigma \times [-1,1]$ satisfying

$$0 < b^p(t) \leq \varphi^{p\prime}(t) \text{ for all } (p,t) \in \Sigma \times [-1,1],$$
$$b^p(t) = \varphi^{p\prime}(t) \text{ for } |t| \geq \epsilon^2.$$

Then the smooth function (in p and t)

$$\varphi_0^p(t) \stackrel{\text{def}}{=} \varphi^p(-1) + \int_{-1}^t b^p(s)\, ds$$

coincides with $\varphi^p(t)$ on $t \leq -\epsilon^2$ and with $\varphi^p(t) - c(p)$ on $t \geq \epsilon^2$ for some $c(p) \geq 0$ which is independent of t and smooth as a function of $p \in \Sigma$. Then set

$$\widetilde{\varphi}^p(t) = \varphi_0^p(t) + c(p)\chi(t),$$

where $\chi : [-1,1] \to [0,1]$ is a smooth non-decreasing function with $\chi(t) \equiv 0$ for $t \leq -\epsilon^2$ and $\chi(t) \equiv 1$ for $t \geq \epsilon^2$. Now $\widetilde{\varphi}^p(t)$ coincides with $\varphi^p(t)$ for t outside of $[-\epsilon^2, \epsilon^2]$, is everywhere smooth and satisfies $\widetilde{\varphi}^{p\prime}(t) \geq b^p(t) > 0$ everywhere.

On $-\epsilon^2 \leq t \leq \epsilon^2$ we have

$$\widetilde{\varphi}^{p\prime}(t) > 0 = \|\widetilde{\gamma}_1^{p\prime}(t)\|;$$

on $|t| \geq \epsilon^2$ we have

$$\widetilde{\varphi}^{p\prime}(t) \geq b^p(t) = \varphi^{p\prime}(t) > \|\widetilde{\gamma}_1^{p\prime}(t)\|.$$

Remark. It is trivial to check that a contact circle defined by a family of paths with $y_3 \equiv 0$ and $y_4 \equiv 0$ on the quadric S is a taut contact circle. The construction in the above proof can also be applied to deform a (smooth) contact circle to one that is taut in the neighbourhood of a given point, where the support of the deformation may be chosen arbitrarily small.

Let Σ be a disc $\{|p| \leq R_0\}$. Then for a positive number $\epsilon < \min(1/2, R_0)$ find a function $a_\epsilon^p(t)$ satisfying

$$0 \leq a_\epsilon^p(t) \leq 1 + 2\epsilon \text{ for all } t \in [-1,1],$$
$$a_\epsilon^p(t) = 0 \text{ for } |p|^2 + t^2 \leq \epsilon^4,$$
$$a_\epsilon^p(t) = 1 \text{ for } |p|^2 + t^2 \geq \epsilon^2,$$
$$\int_0^\epsilon a_\epsilon^p(s)\, ds = \epsilon \text{ for all } p \in \Sigma,$$

and now construct $\widetilde{\gamma}_1^p(t)$ as before, leaving $\varphi^p(t)$ unchanged. We obtain a contact circle which is taut near the point $|p| = t = 0$ and coincides with the old contact circle outside the ball $\{|p|^2 + t^2 \leq \epsilon^2\}$.

In Section 5 we shall use local deformations of contact circles into taut ones, but they will be more direct than the general one desribed here because of the additional geometric character of the contact circles considered there.

4. Extension lemmas

Building on the results from Section 3 we now proceed to prove the extension lemmas.

Proof of Lemma 2.3. Consider $\Sigma \times [0, +\infty)$ and let t be the coordinate on the interval factor. Whenever convenient in this proof, we view forms on Σ (of degree 1 or 2) as forms of the same degree on $\Sigma \times [0, +\infty)$ annihilated by $i(\partial_t)$.

Let $(\omega_1^{(s)}, \omega_2^{(s)})$, $0 \leq s \leq 1$, be a homotopy (via parallelizations) between the parallelizations $(\omega_1^{(0)}, \omega_2^{(0)})$ and $(\omega_1^{(1)}, \omega_2^{(1)})$ of Σ. We look for functions $R(p, s) > 0$ and $\phi(p, s)$ on $\Sigma \times [0, 1]$ such that

$$\omega_1^{(s)} = R \cdot (\cos\phi\, \omega_1^{(0)} - \sin\phi\, \omega_2^{(0)}).$$

We can take $R = 1$ and $\phi = 0$ on the slice $s = 0$. With these as initial values, R and ϕ exist and are smooth on all of $\Sigma \times [0, 1]$ because of the lifting property of covering projections.

Since $(\omega_1^{(0)}, \omega_2^{(0)})$ and $(\omega_1^{(1)}, \omega_2^{(1)})$ are homotopic, they induce the same orientation on Σ.

For a given $\varepsilon = 1$ or -1, we want to induce the parallelization $(\omega_1^{(i)}, \omega_2^{(i)})$ on $\Sigma \times \{i\}$, $i = 0, 1$, by a contact circle whose forms define the same ambient orientation as $\varepsilon dt \wedge \omega_1^{(0)} \wedge \omega_2^{(0)}$. This orientation is realized, in particular, by the following contact circle on $\Sigma \times [0, +\infty)$:

$$\begin{array}{rcrcr}
\tilde{\omega}_1(p, t) & = & \cos(\varepsilon t)\, \omega_1^{(0)}(p) & - & \sin(\varepsilon t)\, \omega_2^{(0)}(p), \\
\tilde{\omega}_2(p, t) & = & \sin(\varepsilon t)\, \omega_1^{(0)}(p) & + & \cos(\varepsilon t)\, \omega_2^{(0)}(p),
\end{array}$$

which has common kernel spanned by ∂_t.

We can modify the function $\phi(p, t)$ by adding to it any integral multiple of 2π. Thus, because of the compactness of Σ, we may assume without loss of generality that ϕ satisfies $\varepsilon\phi \geq 0$ on all of its domain.

Define a graph surface $\Sigma_1 \subset \Sigma \times (0, +\infty)$ by $t = \varepsilon\phi(p, 1) + \pi$. On this surface we have

$$\tilde{\omega}_1|_{t=\varepsilon\phi(p,1)+\pi} = -\frac{1}{R}\omega_1^{(1)}.$$

Since $(\tilde{\omega}_1, \tilde{\omega}_2)$ coincides with $(\omega_1^{(0)}, \omega_2^{(0)})$ on the slice $t = 0$, and since the two given parallelizations define the same orientation, the product $(\tilde{\omega}_1 \wedge \tilde{\omega}_2)(p, t)$ must be a positive multiple of $(\omega_1^{(1)} \wedge \omega_2^{(1)})(p)$ for any value of t. This implies that

$$\tilde{\omega}_2|_{t=\varepsilon\phi(p,1)+\pi} = B\omega_1^{(1)} - C\omega_2^{(1)},$$

with C/R positive. Hence $C = C(p)$ is positive. We now have the following for the bilinear form defined in Section 3,

$$\langle(\tilde{\omega}_1, \tilde{\omega}_2)|_{t=\varepsilon\phi(p,1)+\pi}, (\omega_1^{(1)}, \omega_2^{(1)})(p)\rangle = -\left(C + \frac{1}{R}\right)(\omega_1^{(1)} \wedge \omega_2^{(1)})(p),$$

which is a negative multiple of $(\omega_1^{(1)} \wedge \omega_2^{(1)})(p)$. By Proposition 3.4, if T is any constant with $\Sigma_1 \subset \Sigma \times [0, T)$, we can find a contact circle defined between Σ_1 and $\Sigma \times \{T\}$ that defines the given ambient orientation, whose common kernel is spanned by ∂_t, and which induces $(\tilde{\omega}_1, \tilde{\omega}_2)$ on Σ_1 and $(\omega_1^{(1)}, \omega_2^{(1)})$ on $\Sigma \times \{T\}$.

Applying Lemma 2.2 to the resulting piecewise smooth contact circle on $\Sigma \times [0, T]$ yields a smooth contact circle on the same domain which induces $(\omega_1^{(0)}, \omega_2^{(0)})$ on the slice $t = 0$ and $(\omega_1^{(1)}, \omega_2^{(1)})$ on the slice $t = T$. This contact circle defines the given ambient orientation and has common kernel spanned by ∂_t. Finally, rescale $[0, T]$ down to $[0, 1]$.

Proof of Lemma 2.4. Let (α_1, α_2) be a pair of pointwise linearly independent 1-forms on (a neighbourhood of the boundary of) the solid torus $S^1 \times D^2$, with common kernel transverse to the boundary $T^2 = S^1 \times \partial D^2$. Homotopically, the meridian $* \times S^1$ is well-defined, and we fix a longitude $S^1 \times *$. This determines the Lie group framing of T^2 up to orientation, which we choose so as to coincide with the orientation induced by (α_1, α_2). We can then measure along the meridian the rotation number p_1 of the parallelization induced by (α_1, α_2) with respect to the Lie group framing, and the rotation number p_2 along the longitude. For short, we say (α_1, α_2) has rotation number (p_1, p_2).

By Lemmas 2.2 and 2.3 it suffices for the proof of Lemma 2.4 to show that we can realize any rotation number $(2p + 1, q)$ by a contact circle on $S^1 \times D^2$ which induces given orientations on $S^1 \times D^2$ and T^2. We claim that this problem can be reduced to the following lemma.

Lemma 4.1. (i) *Fix an orientation on the solid torus $S^1 \times D^2$ and its boundary. There is a contact circle on the solid torus inducing the given orientations with kernel transverse to the boundary and rotation number* 1 *along the meridian.*

(ii) *Let D be a closed disc and let $D_0, \dots, D_p \subset \mathrm{interior}(D)$ be disjoint closed discs. Let $D(p) \stackrel{\mathrm{def}}{=} D - \mathrm{interior}(D_0 \cup \dots \cup D_p)$. For $n \geq 3$ there is a contact circle, defined on the compact region $S^1 \times D(p)$ in $S^1 \times D$, with kernel transverse to the boundary of $S^1 \times D(p)$, rotation number $(2p+1, -n)$ on $S^1 \times \partial D$, and rotation number* 1 *along the meridian of $S^1 \times \partial D_k$, $k = 0, \dots, p$.*

Indeed, given a contact circle with rotation number $(1, q_0)$ and inducing the right orientations, whose existence is guaranteed by (i), we can realize any other rotation number $(1, q)$ by changing the choice of longitude (which can be effected by an orientation preserving diffeomorphism of $S^1 \times D^2$). We can then use (ii) and Lemmas 2.2 and 2.3 to realize any rotation number $(2p+1, -n)$, $p \geq 0, n \geq 3$, by a contact circle on $S^1 \times D^2$.

Further we can realize any rotation number $(2p+1, q)$, $p \geq 0$, by first realizing $(2p+1, q - k(2p+1))$ with $k \in \mathbb{N}$ sufficiently large so that $q - k(2p+1) \leq -3$, and then changing the longitude by adding k times the meridian of $S^1 \times D^2$.

The rotation number $(-(2p+1), q)$, $p \geq 0$, can be obtained by first realizing $(2p+1, -q)$ and then applying an orientation preserving diffeomorphism of $S^1 \times D^2$ which reverses both meridian and longitude.

Finally, we have to check that any choice of orientation can be induced by a suitable contact circle. The previous arguments show that any rotation number $(2p+1, q)$ can be realized by a contact circle (ω_1, ω_2) with one particular choice of orientations. To change the ambient orientation (and the orientation of T^2), we start with (ω_1, ω_2) having rotation number $(-(2p+1), q)$ and pull back this contact circle by an orientation reversing diffeomorphism of $S^1 \times D^2$ which reverses the meridian and preserves the longitude.

To obtain rotation number $(2p+1, q)$ with the opposite orientation on T^2 but same ambient orientation on $S^1 \times D^2$, we choose (ω_1, ω_2) realizing $(-(2p+1), -q)$ and then change to $(\omega_1, -\omega_2)$. This concludes the proof of Lemma 2.4.

Proof of Lemma 4.1. (i) Start with the contact circle

$$\begin{aligned}\omega_1^0 &= \cos\theta\, dx - \sin\theta\, dy,\\ \omega_2^0 &= \sin\theta\, dx + \cos\theta\, dy,\end{aligned}$$

on $S^1 \times D^2$, where θ denotes the S^1-coordinate, and x, y cartesian coordinates for D^2. The ambient orientation is defined by the volume form

$$V = \omega_1^0 \wedge d\omega_1^0 = dx \wedge dy \wedge d\theta,$$

and we define an orientation on the common kernel by calling the vector field X with $i(X)V = \omega_1^0 \wedge \omega_2^0$ a *positive* section of the common kernel. Here we find $X = \partial_\theta$. Perturbing the contact circle as in the proof of Lemma 2.5 so as to make the common kernel point outwards yields a contact circle (ω_1, ω_2) with rotation number $+1$ along the meridian, where we give the meridian the positive orientation in the xy-plane. Passing from (ω_1, ω_2) to $(\omega_1, -\omega_2)$ changes the boundary orientation (and makes the common kernel point inwards) and gives rotation number -1 with respect to this orientation.

If we start again with (ω_1^0, ω_2^0) and make the common kernel point inwards, we realize the second boundary orientation and rotation number $+1$; $(\omega_1, -\omega_2)$ realizes the first orientation and rotation number -1. Replacing θ by $-\theta$ yields the opposite ambient orientation.

Henceforth we disregard all questions of orientation. We prepare the ground for the proof of (ii) by first giving an alternative proof of (i) which avoids the use of deformations.

We start with 1-forms α, β on the solid torus which are everywhere linearly independent and have common kernel transverse to the boundary. For any function f with values in $\mathbb{R}/2\pi\mathbb{Z}$ the pair of forms

$$\begin{aligned}\omega_1 &= \cos(f)\, \alpha - \sin(f)\, \beta,\\ \omega_2 &= \sin(f)\, \alpha + \cos(f)\, \beta,\end{aligned}$$

has the same common kernel as (α, β). We look for a condition on f which ensures that (ω_1, ω_2) is a contact circle.

Given constants λ_1, λ_2, consider $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$. If we define the functions c_1, c_2 by

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \cos f & \sin f \\ -\sin f & \cos f \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix},$$

then we have $\omega = c_1\alpha + c_2\beta$ and we compute

$$\begin{aligned}(d\omega)\wedge\omega =& d\left(\begin{bmatrix} c_1 & c_2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right)\wedge\begin{bmatrix} \alpha & \beta \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\\ =& \begin{bmatrix} c_1 & c_2 \end{bmatrix}\begin{bmatrix} \alpha\wedge d\alpha & \alpha\wedge d\beta \\ \beta\wedge d\alpha & \beta\wedge d\beta \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\\ &+\begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}\left(d\begin{bmatrix} \cos f & -\sin f \\ \sin f & \cos f \end{bmatrix}\right)\wedge\begin{bmatrix} \alpha \\ \beta \end{bmatrix}\\ &\wedge\begin{bmatrix} \alpha & \beta \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\\ =& \begin{bmatrix} c_1 & c_2 \end{bmatrix} S^{(0)}_{\alpha\beta}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + (\lambda_1^2+\lambda_2^2)df\wedge\alpha\wedge\beta,\end{aligned}$$

where $S^{(0)}_{\alpha\beta}$ is the symmetric part of the matrix

$$\begin{bmatrix} \alpha\wedge d\alpha & \alpha\wedge d\beta \\ \beta\wedge d\alpha & \beta\wedge d\beta \end{bmatrix}.$$

Let V be a fixed volume form on the solid torus, and define the vector field X by the identity $\alpha\wedge\beta = i(X)V$. Then

$$\omega\wedge d\omega = \left(\begin{bmatrix} c_1 & c_2 \end{bmatrix} S_{\alpha\beta}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + (\lambda_1^2+\lambda_2^2)(Xf)\right)V\ ,$$

where $S_{\alpha\beta}$ is the symmetric scalar matrix such that the entries of $S^{(0)}_{\alpha\beta}$ equal V multiplied by the entries of $S_{\alpha\beta}$.

The number

$$\begin{bmatrix} c_1 & c_2 \end{bmatrix} S_{\alpha\beta}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

is bounded in absolute value by $(\lambda_1^2+\lambda_2^2)|S_{\alpha\beta}|$, where $|S_{\alpha\beta}|$ is the spectral radius of $S_{\alpha\beta}$. Therefore the pointwise inequality $|Xf| > |S_{\alpha\beta}|$ is a sufficient condition for (ω_1,ω_2) to be a contact circle. Since $S_{\alpha\beta}$ is independent of f, we shall specify α and β first and then look for f with $Xf > |S_{\alpha\beta}|$.

Let now x, y, θ be coordinates on $S^1\times D^2$ as before and set $V = dx\wedge dy\wedge d\theta$. For the particular choice

$$\begin{aligned}\alpha &= -dx + x\,d\theta,\\ \beta &= dy - y\,d\theta,\end{aligned}$$

the symmetric matrix $S_{\alpha\beta}$ is zero and $X = -\partial_\theta - x\partial_x - y\partial_y$. Thus we get a contact circle with common kernel transverse to the boundary if we set $f = -\theta$.

We now study the rotation number of this contact circle. Let $j : S^1 \times \partial D^2 \to S^1 \times D^2$ be the inclusion, and let ψ be the angle coordinate along ∂D^2. Then

$$\begin{aligned} j^*\alpha &= \sin\psi\, d\psi + \cos\psi\, d\theta, \\ j^*\beta &= \cos\psi\, d\psi - \sin\psi\, d\theta. \end{aligned}$$

Along the longitude $\{\psi = 0\}$ we have $j^*\alpha = d\theta$ and $j^*\beta = d\psi$, thus $(d\theta, d\psi)$ is a Lie group framing on $S^1 \times \partial D^2$ defining the same orientation as $(j^*\alpha, j^*\beta)$. We conclude that (α, β) has rotation number $(1,0)$, and (ω_1, ω_2) has rotation number $(1,1)$.

(ii) Here we consider

$$\begin{aligned} \alpha &= -dx + x\sin y\, dy - x\cos y\, d\theta, \\ \beta &= \cos y\, dy + \sin y\, d\theta. \end{aligned}$$

Then $\alpha \wedge \beta = i(X)V$ with

$$X = -\cos y\partial_\theta + x\partial_x + \sin y\partial_y,$$

which is nowhere zero. Thus α and β are everywhere linearly independent. We now construct the desired manifold $S^1 \times D(p)$ with boundary transverse to X.

Let X_0 denote the projection $x\partial_x + \sin y\partial_y$ of X into the xy-plane. The function $h(x,y) = x^2 + \tan^2(y/2)$ is infinite where y is an odd multiple of π and is finite and smooth everywhere else. Moreover, h satisfies the identity $X_0 h = 2h$, and its zeros are the points $(x,y) = (0, 2k\pi)$ with $k \in \mathbb{Z}$. For any positive real number a the inequality $h(x,y) \leq a$ defines a planar region which is a disjoint union of topological discs centered at the zeros of h.

For $k \in \mathbb{Z}$ let $D_k(a)$ be the connected component of $\{h(x,y) \leq a\}$ containing the point $(0, 2k\pi)$. Let $D_k^+(a)$ (resp. $D_k^-(a)$) be the part of $D_k(a)$ determined by $y \geq 2k\pi$ (resp. $y \leq 2k\pi$).

Fix a positive integer p and let

$$D = D_0(1)^- \cup ([-1,1] \times [0, 2p\pi]) \cup D_p^+(1).$$

This is topologically a disc with C^1-boundary (Fig. 5). The vector field X_0 has $2p+1$ zeros in D, namely, the points $(0, 2k\pi)$, $k = 0, \ldots, p$, which are sources, and the points $(0,(2k-1)\pi)$, $k=1,\ldots,p$, which are saddles. The stable separatrices are straight segments on the y-axis, while the unstable separatrices are half lines parallel to the x-axis.

The boundary ∂D consists of part of the lines $x=\pm 1$ and part of the curves $h(x,y)=1$. Then $X_0x=1$ along the straight part and $X_0h=2$ along the curved part, which means that X_0 is transverse to the boundary of D. Hence X is transverse to the boundary of the solid torus $S^1\times D$. The disc D contains in its interior the discs $D_k=D_k(1/2)$, $k=0,\ldots,p$, and we define $D(p)$ as D with the interiors of the D_k removed. Along the boundary of D_k we have $X_0h=1$, thus X_0 is transverse to these boundaries. We conclude that X_0 is transverse to $\partial D(p)$ and X is tranverse to the boundary of $S^1\times D(p)$.

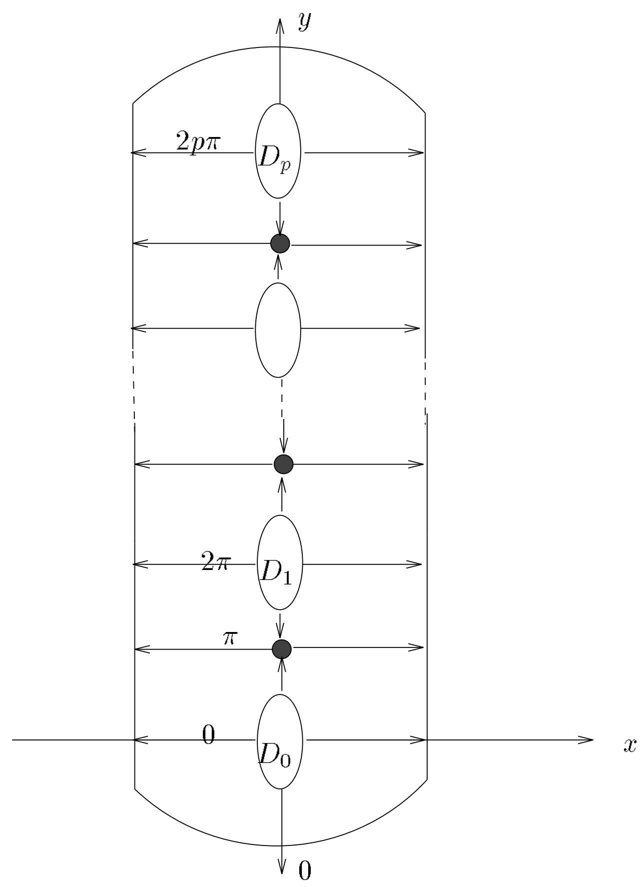

FIGURE 5. The domain $D(p)$ and vector field X_0.

Now we need a function f on $S^1 \times D(p)$ with values in $\mathbb{R}/2\pi\mathbb{Z}$ and such that $Xf > |S_{\alpha\beta}|$. We make the ansatz

$$f(x, y, \theta) = f_{n,b,c}(x, y, \theta) = n\theta + b \cdot c(h(x, y)),$$

where n is a positive integer, b, a positive constant, and $c(s)$, a smooth function of the real variable s satisfying

$$\begin{aligned} &\text{(i)} && c(s) = s \ \text{ for } \ s \leq 4 + 2\sqrt{2}, \\ &\text{(ii)} && c'(s) \geq 0 \ \text{ for all } \ s, \\ &\text{(iii)} && c(s) \ \text{ is constant for } \ s \geq 8. \end{aligned}$$

Condition (iii) implies that $c(h(x, y))$ extends as a smooth, real-valued function over the whole xy-plane, hence f is well-defined. We next show that

$$Xf \geq \min \left(\frac{n}{\sqrt{2}}, b - n \right)$$

everywhere on $S^1 \times D(p)$. We compute

$$Xf = -n \cos y + 2b \cdot h \cdot c'(h),$$

and so $Xf \geq -n \cos y$ because of condition (ii). The inequality $-\cos y \geq 1/\sqrt{2}$ defines a domain $S^1 \times F$ where F is a disjoint union of stripes in the xy-plane, parallel to the x-axis. We have $Xf \geq n/\sqrt{2}$ on $S^1 \times F$. The minimum of $h(x, y)$ on $D(p)$ is $1/2$ by the definition of $D(p)$; the maximum of $h(x, y)$ on $D(p) - F$ is

$$1 + \tan^2(\frac{\pi}{2} - \frac{\pi}{8}) = 4 + 2\sqrt{2}.$$

So by condition (i) we have $c'(h) \equiv 1$ on $D(p) - F$. Consequently

$$Xf \geq -n + 2b \cdot (1/2) \cdot 1 = b - n \ \text{ on } \ S^1 \times (D(p) - F).$$

The claimed inequality follows.

Thus (ω_1, ω_2) will be a contact circle on $S^1 \times D(p)$ if $n/\sqrt{2} > |S_{\alpha\beta}|$ and $b - n > |S_{\alpha\beta}|$. For the above choice of α, β, and V, we compute

$$S_{\alpha\beta} = \begin{bmatrix} -x \sin y & \sin^2 \frac{y}{2} \\ \sin^2 \frac{y}{2} & 0 \end{bmatrix},$$

whose eigenvalues are

$$\lambda = -\frac{x}{2}\sin y \pm \sin\frac{y}{2}\sqrt{x^2\cos^2\frac{y}{2} + \sin^2\frac{y}{2}}\ ,$$

so obviously $|\lambda| \leq 3/2$ on $S^1 \times D(p)$. Hence $n \geq 3$ and $b \geq n+2$ will do, that is, we get a contact circle on $S^1 \times D(p)$ if we set

$$f = n\theta + (n+2)\cdot c(h(x,y))$$

for any integer $n \geq 3$. In fact, one can easily show with a precise estimate on $|\lambda|$ that $n \geq 2$ is sufficient.

It remains to check that (ω_1, ω_2) has the claimed rotation numbers. Since the function $(n+2)\cdot c(h(x,y))$ is $\mathbb{R}$-valued, the rotation numbers of (ω_1,ω_2) are the same as for the pair (ω_1', ω_2') defined like (ω_1,ω_2) but with $f = n\theta$. Let ψ be a circular coordinate along ∂D which equals y on $\{x=1\}\cap\partial D$. Let j be the inclusion of $S^1\times\partial D$ into $S^1 \times D$. Along the longitude determined by $x=1$ and $y=0$ we have $j^*\alpha = -d\theta$ and $j^*\beta = d\psi$, hence $(d\psi, d\theta)$ is a Lie group framing of $S^1\times\partial D$ defining the same orientation as $(j^*\alpha, j^*\beta)$. Also, along the longitude we have

$$j^*\omega_1' = -\cos(n\theta)\, d\theta - \sin(n\theta)\, d\psi,$$

so the rotation number of (ω_1,ω_2) along the longitude is $-n$.

The pairs (ω_1', ω_2') and (α,β) coincide along the slice $\{\theta = 0\}$ of the solid torus, so they have the same rotation numbers along the meridians. Let now j_k be the inclusion of $S^1 \times \partial D_k$ into $S^1\times D$ and let ψ_k be a circular coordinate along ∂D_k. At the point $(x,y,t) = (1/\sqrt{2}, 2k\pi, 0)$ the form $j_k^*\alpha$ equals $-(1/\sqrt{2})dt$, and $j_k^*\beta$ equals a positive multiple of $d\psi_k$, hence $(d\psi_k, d\theta)$ is a Lie group framing of $S^1\times\partial D_k$ defining the same orientation as $(j_k^*\alpha, j_k^*\beta)$.

For the final step in the proof of Lemma 4.1 we need the following lemma.

Lemma 4.2. *Let Σ be a disc in $\mathbb{R}^2$, and X_0 a vector field on Σ transverse to $\partial\Sigma$ with exactly $2p+1$ zeros inside Σ, of which $p+1$ are sources and p are saddles. Let $A(x,y)$ be a function on Σ which is positive at the saddles and negative at the sources. Consider $X = A\partial_t + X_0$ as a vector field on the solid cylinder $\mathbb{R}\times\Sigma$, where t is the coordinate along the $\mathbb{R}$-factor. Suppose that (α,β) is a pair of pointwise linearly independent 1-forms on $\mathbb{R}\times\Sigma$. Assume further that $i(X)(\alpha\wedge\beta)\equiv 0$ and choose a circular coordinate σ along $\partial\Sigma$ such that (α,β) induces the same orientation as $(d\sigma, dt)$ on $\mathbb{R}\times\partial\Sigma$.*

If we orient the meridian of the solid cylinder in the direction of increasing σ, then (α, β) has rotation number $2p+1$ along this meridian.

Applying Lemma 4.2 with $A(x,y) = -\cos y$, $X_0 = x\partial_x + \sin y\,\partial_y$ and $\Sigma = D$ we conclude that (α, β) has rotation number $2p+1$ along the meridian of $S^1 \times D$. With $\Sigma = D_k$ we see that (α, β) has rotation number 1 along the meridian of $S^1 \times D_k$. This completes the proof of Lemma 4.1.

Proof of Lemma 4.2. We may assume that Σ is the unit disc $\{x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2 \subset \mathbb{R}^3$. Consider the hemisphere

$$\Sigma_+ = \{(x,y,t) \mid x^2 + y^2 + t^2 = 1 \ , \ t \geq 0\}$$

and the piecewise smooth sphere $\Theta = (\Sigma \times \{0\}) \cup \Sigma_+$. There is a continuous (indeed, piecewise smooth) bundle $\mathcal{F} \to \Theta$ which on $\Sigma \times \{0\}$ is a plane field complementary to X, and on Σ_+ coincides with $T\Sigma_+$. Define a Gauß map $\nu : \Theta \to S^2$ by assigning to $p \in \Sigma_+$ the outward unit normal $\nu(p)$, and to $p \in \Sigma \times \{0\}$ the vector $\nu(p)$ orthogonal to $\mathcal{F}_p$ and on the same side of $\mathcal{F}_p$ as $X(p)$. Here the choice of Riemannian metric on $\mathbb{R}^3$ is irrelevant. By the hypotheses on X_0 and $A(x,y)$, this map ν has degree $p+1$.

Any choice of a Riemannian metric on $\mathcal{F}$ allows us to pass from 1-forms to vector fields. So we may view α as a nonvanishing section of $\mathcal{F}$ over Σ and extend it to a global section over Θ in general position (i.e., with transverse intersection with the zero section). On Σ_+ the extended α is a tangent vector field, and if we now identify this hemisphere with a disc D_+ then the vector field must have total index $2(p+1)$ in this disc.

The cylinder $\mathbb{R} \times \partial\Sigma$ and the hemisphere Σ_+ are tangent along the circle $\partial\Sigma \times \{0\}$, and so the framing $(d\sigma, dt)$ becomes a framing of the hemisphere along its equator. After the identification of the hemisphere with the disc D_+ this framing (now viewed as a pair of vector fields) is $\{\partial_\sigma, \nu_{\text{in}}\}$, where ν_{in} is the normal along ∂D_+ pointing towards the inside of D_+.

With ∂D_+ oriented in the direction of increasing σ, the vector field $\alpha|_{\partial D_+}$ has rotation number $2p+2$ with respect to any global framing of D_+ defining the same orientation as $(\partial_\sigma, \nu_{\text{in}})$. Since $\{\partial_\sigma, \nu_{\text{in}}\}$ has rotation number 1 with respect to any such global framing, the rotation number of α along ∂D_+ with respect to the moving frame $(\partial_\sigma, \nu_{\text{in}})$ must be $(2p+2) - 1 = 2p+1$.

5. Contact circles on geometric manifolds

We now give explicit global descriptions of contact circles on certain geometric manifolds. In particular, we shall provide a proof of the following theorem. As far as the existence of contact circles is concerned, this theorem is contained in Theorem 1.2, but the statement about tight contact structures can only be deduced from the explicit constructions in this section.

Theorem 5.1. *Suppose the closed* 3*-manifold* M *is the connected sum of any number of copies of the following manifolds:*

(i) *All the manifolds listed in Theorem* 1.3*.*

(ii) *Orientable* T^2*-bundles over* S^1*.*

(iii) *The orientable manifolds modelled on the geometry* $S^2 \times E^1$*, namely,* $S^2 \times S^1$ *and* $\mathbb{R}P^3 \# \mathbb{R}P^3$*.*

Then M *admits a contact circle* (ω_1, ω_2) *such that the contact structure* $\ker \omega_1$ (*and hence* $\ker(\lambda_1\omega_1 + \lambda_2\omega_2)$ *for any* $(\lambda_1, \lambda_2) \in S^1$) *is tight.*

Remarks. (1) $S^3, \widetilde{\mathrm{SL}}_2, \widetilde{\mathrm{E}}_2$ and $S^2 \times E^1$ are four of the eight 3-dimensional Thurston geometries [16]. The T^2-bundles over S^1 with periodic monodromy $A \in \mathrm{SL}_2\mathbb{Z}$ are the left-quotients of $\widetilde{\mathrm{E}}_2$. The T^2-bundles over S^1 with non-periodic monodromy A satisfying $|\mathrm{trace}\, A| = 2$ are modelled on the geometry Nil^3. The T^2-bundles with monodromy satisfying $|\mathrm{trace}\, A| \geq 3$ are modelled on the geometry Sol^3. A detailed description of the geometries Nil^3, Sol^3 and $S^2 \times E^1$ and a description of T^2-bundles over S^1 with $|\mathrm{trace}\, A| \geq 2$ as quotients of Nil^3 and Sol^3 will be given below. Note that all the manifolds in (i) to (iii), except for $\mathbb{R}P^3 \# \mathbb{R}P^3$, are indecomposable.

(2) In [6] it was shown that if ω_1 is part of a taut contact circle (ω_1, ω_2), then the contact structure $\ker \omega_1$ is tight. See [2], [9] as well as Section 5.6 below for the definition and significance of *tight* contact structures. As we shall see in the next section, at least on open manifolds there are contact circles containing overtwisted contact structures.

The proof of Theorem 5.1 is organized as follows. We construct contact circles on each of the manifolds listed in Theorem 5.1 (ii) and (iii) and show that locally these contact circles can be made taut. For (i) we refer the reader to [6]. In Section 5.1 we deal with left-quotients of the Lie group Nil^3, in Section 5.2 with left-quotients of Sol^3, and in Section 5.3 we collect this information to prove part (ii) of Theorem 5.1.

In Section 5.4 we deal with the geometry $S^2 \times E^1$. In Section 5.5 we show that one can attach 1-handles (and in particular form connected sums) near points where the contact circle is taut and extend the contact circle over the 1-handle. The statement in Theorem 5.1 on tightness will be proved in Section 5.6.

5.1. The Heisenberg group

The Heisenberg group Nil^3 is the nilpotent Lie group of matrices of the form

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

So we can think of Nil^3 as $\mathbb{R}^3$ with multiplication

$$(x_0, y_0, z_0)\,(x, y, z) = (x_0 + x, y_0 + y, z_0 + z + x_0 y).$$

According to Scott [16], every compact left-quotient of Nil^3 is diffeomorphic to one of the form $\Gamma_k \setminus Nil^3$, $k \in \mathbb{Z} \setminus \{0\}$, where the discrete subgroup Γ_k of Nil^3 is the lattice spanned by the elements

$$(k, 0, 0),\ (0, 1, 0),\ (0, 0, 1).$$

It is easy to see that these left-quotients are precisely the T^2-bundles over S^1 with monodromy $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, where the bundle projection is induced by the map

$$(x, y, z) \longmapsto y.$$

Proposition 5.2. *Let $M_k = \Gamma_k \setminus Nil^3$. On Nil^3, set*

$$\omega_1 = \cos(2\pi y)\, dx - \sin(2\pi y)\,(dz - f(x)\, dy),$$

$$\omega_2 = \sin(2\pi y)\, dx + \cos(2\pi y)\,(dz - f(x)\, dy),$$

where f is a smooth, monotone increasing function that satisfies

$$\begin{array}{lcll} f(x) & = & x & \textit{near } x = 0, \\ f(x) & \equiv & k/2 & \textit{near } x = k/2, \\ f(x+k) & = & f(x) + k & \textit{for all } x. \end{array}$$

Then (ω_1,ω_2) *defines a contact circle on* Nil^3 *that is invariant under* Γ_k *and hence descends to* M_k*. This contact circle is taut near* $x\in(k/2)+k\mathbb{Z}$*.*

Moreover, all the closed orbits of the flow of $\xi_{\lambda_1,\lambda_2}$ *are non-contractible, where* $\xi_{\lambda_1,\lambda_2}$ *denotes the Reeb vector field of* $\lambda_1\omega_1+\lambda_2\omega_2$*.*

Remark. The statement on closed orbits implies (by Theorem 1 of [11]) that the contact structures $\ker(\lambda_1\omega_1+\lambda_2\omega_2)$ are tight. However, in Section 5.6 we shall give a more elementary proof of this fact.

Proof. It is a straightforward check that the forms ω_1,ω_2 are invariant under Γ_k.

Given $(\lambda_1,\lambda_2)\in S^1$, write $\lambda_1=\cos\theta$ and $\lambda_2=\sin\theta$ with $\theta\in[0,2\pi)$. Then

$$\omega_{\lambda_1,\lambda_2}=\lambda_1\omega_1+\lambda_2\omega_2=C\,dx-S\,(dz-f(x)dy),$$

where

$$C=\cos(2\pi y-\theta)\quad\text{and}\quad S=\sin(2\pi y-\theta).$$

One easily computes

$$\begin{aligned} d\omega_{\lambda_1,\lambda_2} &= (2\pi+f')S\,dx\wedge dy-2\pi C\,dy\wedge dz,\\ \omega_{\lambda_1,\lambda_2}\wedge d\omega_{\lambda_1,\lambda_2} &= -(2\pi+S^2f')\,dx\wedge dy\wedge dz. \end{aligned}$$

This shows that (ω_1,ω_2) is a contact circle which is taut near $x\in(k/2)+k\mathbb{Z}$, where $f'=0$, and that the Reeb vector field $\xi_{\lambda_1,\lambda_2}$ is proportional to

$$2\pi C\partial_x-(2\pi+f')S\partial_z,$$

hence the flow of $\xi_{\lambda_1,\lambda_2}$ is tangent to the fibres of the torus fibration $M_k\to S^1$. Moreover, on a fixed fibre F_y, the coefficient function of at least one of ∂_x and ∂_z is nowhere zero. Hence, any closed orbit of $\xi_{\lambda_1,\lambda_2}$ in the fibre F_y represents a non-trivial element in $\pi_1(F_y)$.

The fundamental group $\pi_1(M_k)$ is generated by the elements α,σ,τ, where

$$\begin{aligned} \alpha(u) &= (0,u,0), & u\in[0,1],\\ \sigma(u) &= (ku,0,0), & u\in[0,1],\\ \tau(u) &= (0,0,u), & u\in[0,1]. \end{aligned}$$

Here we use the same symbol for a loop in M_k and the corresponding element in $\pi_1(M_k)$. The generating relations in $\pi_1(M_k)$ are

$$\sigma\tau = \tau\sigma, \quad \alpha\tau = \tau\alpha, \quad \alpha^{-1}\sigma\alpha = \sigma\tau^k,$$

where we compose elements in $\pi_1(M_k)$ from the right. A closed orbit of the flow of $\xi_{\lambda_1,\lambda_2}$ represents an element of the form $\sigma^p\tau^q$, with $(p,q) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, which is non-trivial in $\pi_1(M_k)$.

Remark. We can define another contact circle on M_k by setting

$$\omega_1' = \cos(2\pi x)\,dy - \sin(2\pi x)\,(dz - f(x)\,dy),$$

$$\omega_2' = \sin(2\pi x)\,dy + \cos(2\pi x)\,(dz - f(x)\,dy).$$

If we consider the fibration $M_k \to S^1$ induced by the map $(x,y,z) \mapsto x$, which also gives M_k the structure of a T^2-bundle with monodromy $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, the statement and proof of Proposition 5.2 remain virtually identical with (ω_1,ω_2) replaced by (ω_1',ω_2').

5.2. The Lorentz group

The inhomogeneous Lorentz group Sol^3 is the solvable Lie group defined as a split extension of $\mathbb{R}^2$ by $\mathbb{R}$,

$$0 \longrightarrow \mathbb{R}^2 \longrightarrow Sol^3 \longrightarrow \mathbb{R} \longrightarrow 0,$$

where an element z in the quotient $\mathbb{R}$ acts on $\mathbb{R}^2$ by sending (x,y) to $(e^z x, e^{-z}y)$. Thus, Sol^3 can be identified with $\mathbb{R}^3$ with multiplication

$$(x_0,y_0,z_0)\,(x,y,z) = (x_0 + e^{z_0}x, y_0 + e^{-z_0}y, z_0 + z).$$

The compact left-quotients of Sol^3 are precisely the T^2-bundles over S^1 with orientation preserving hyperbolic glueing map of positive trace (cf. [16]), where the bundle projection is given by

$$(x,y,z) \longmapsto z,$$

that is, induced by the projection $Sol^3 \to \mathbb{R}$. This means that the glueing map is given by a diffeomorphism $T^2 \to T^2$ which is covered by a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ represented by the matrix $A' = \begin{pmatrix} e^{\gamma} & 0 \\ 0 & e^{-\gamma} \end{pmatrix}$

with respect to a suitable basis, where $\gamma \neq 0$ and trace $A' \in \mathbb{Z}$ (since A' is conjugate to the monodromy matrix $A \in \mathrm{SL}_2\mathbb{Z}$). This is easily seen to be equivalent to A satisfying trace $A \geq 3$.

So any compact left-quotient of Sol^3 is of the form $\Gamma \setminus Sol^3$, where the lattice Γ in Sol^3 is generated by elements of the form

$$(\alpha_1, \beta_1, 0),\ (\alpha_2, \beta_2, 0),\ (0, 0, \gamma),$$

where (α_1, β_1), (α_2, β_2) span a lattice of translations in $\mathbb{R}^2$, and $\gamma > 0$.

Proposition 5.3. *Let M be a compact left-quotient of Sol^3 under a lattice Γ as above. On Sol^3, set*

$$\omega_1 = \cos\left(\frac{2\pi n}{\gamma} z\right) e^{-g(z)} dx - \sin\left(\frac{2\pi n}{\gamma} z\right) e^{g(z)} dy,$$

$$\omega_2 = \sin\left(\frac{2\pi n}{\gamma} z\right) e^{-g(z)} dx + \cos\left(\frac{2\pi n}{\gamma} z\right) e^{g(z)} dy,$$

where $n \in \mathbb{N}$ and g is a smooth, monotone increasing function that satisfies

$$\begin{array}{lcll} g(z) & = & z & \textit{near } z = 0, \\ g(z) & \equiv & \gamma/2 & \textit{near } z = \gamma/2, \\ g(z+\gamma) & = & g(z) + \gamma & \textit{for all } z. \end{array}$$

If n is chosen sufficiently large (the proof will show that we need $2\pi n/\gamma - g' > 0$), then (ω_1, ω_2) defines a contact circle on Sol^3 that is invariant under Γ, so it descends to M. This contact circle is taut near $z \in \gamma/2 + \gamma\mathbb{Z}$.

Furthermore, all the closed orbits of the Reeb vector field $\xi_{\lambda_1,\lambda_2}$ of $\lambda_1\omega_1 + \lambda_2\omega_2$ are non-contractible.

Proof. As in the case of the Heisenberg group it is easy to check that the forms ω_1, ω_2 are invariant under Γ.

Again write $\lambda_1 = \cos\theta$ and $\lambda_2 = \sin\theta$ with $\theta \in [0, 2\pi)$ and define

$$C = \cos\left(\frac{2\pi n}{\gamma} z - \theta\right) \quad \text{and} \quad S = \sin\left(\frac{2\pi n}{\gamma} z - \theta\right).$$

Then

$$\omega_{\lambda_1,\lambda_2} = C e^{-g(z)} dx - S e^{g(z)} dy,$$

and we compute

$$\begin{aligned} d\omega_{\lambda_1,\lambda_2} &= -\left(\frac{2\pi n}{\gamma}S + Cg'\right) e^{-g} dz \wedge dx \\ &\quad + \left(\frac{2\pi n}{\gamma}C + Sg'\right) e^{g} dy \wedge dz, \\ \omega_{\lambda_1,\lambda_2} \wedge d\omega_{\lambda_1,\lambda_2} &= \left(\frac{2\pi n}{\gamma} + 2SCg'\right) dx \wedge dy \wedge dz. \end{aligned}$$

Since $2SC$ is the sine of twice the angle, the condition for (ω_1, ω_2) to be a contact circle is $g' < 2\pi n/\gamma$. But g' is periodic and thus bounded, and so the condition is satisfied for n sufficiently large.

Near $z \in \gamma/2 + \gamma\mathbb{Z}$ we have $g' = 0$ and hence a taut contact circle.

We also see that $\xi_{\lambda_1,\lambda_2}$ is proportional to

$$\left(\frac{2\pi n}{\gamma}C + Sg'\right) e^{g} \partial_x - \left(\frac{2\pi n}{\gamma}S + Cg'\right) e^{-g} \partial_y.$$

As in the Nil^3 case, this implies that the flow of $\xi_{\lambda_1,\lambda_2}$ is along the fibres of the torus fibration $M \to S^1$.

The rest of the argument is now completely analogous to the proof of Proposition 5.2.

5.3. Torus bundles over S^1

With the results from the two preceding sections we are now going to prove part (ii) of Theorem 5.1. Suppose M is an orientable T^2-bundle over S^1 with monodromy $A \in \mathrm{SL}_2\mathbb{Z}$. If $\operatorname{trace} A \geq 3$, we have seen in Section 3 that M is a left-quotient of Sol^3, so this case of Theorem 5.1 follows from Proposition 5.3. If $\operatorname{trace} A \leq -3$, then M is diffeomorphic to a quotient of the form $\Gamma \setminus Sol^3$, where Γ is a discrete subgroup of the isometry group of Sol^3 generated by elements $(\alpha_1, \beta_1, 0), (\alpha_2, \beta_2, 0) \in Sol^3$ (as in the case when $\operatorname{trace} A \geq 3$) and a generator

$$(x, y, z) \longmapsto (-e^{\gamma}x, -e^{-\gamma}y, z + \gamma).$$

This follows from [18]; see also [4]. The forms ω_1, ω_2 in Proposition 5.3 are preserved only up to sign by this map, but if we replace $2\pi nz/\gamma$ by $n\pi z/\gamma$ with n odd in the argument of cos and sin, we obtain a contact circle that descends to $\Gamma \setminus Sol^3$.

If $|\operatorname{trace} A| \leq 2$, then (as shown in [15]) either A is a periodic matrix or it is conjugate to $\pm A_k = \pm \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, $k \in \mathbb{Z} \setminus \{0\}$.

Up to conjugation, there are exactly five periodic matrices in $\mathrm{SL}_2\mathbb{Z}$, and the corresponding T^2-bundles over S^1 are precisely the left-quotients of $\widetilde{\mathrm{E}}_2$ (cf. [6]), so this case of Theorem 5.1 is covered by Theorem 1.3 and the result from [6] that any taut contact circle consists of tight contact structures.

If A is conjugate to A_k, then we have seen in Section 5.1 that M is a left-quotient of Nil^3, so we can apply Proposition 5.2.

Finally, if A is conjugate to $-A_k$, then M is diffeomorphic to $\Gamma\backslash Nil^3$, where Γ is a discrete subgroup of the isometry group of Nil^3 generated by the elements $(k,0,0),(0,0,1)\in Nil^3$ and the map

$$(x,y,z)\longmapsto(-x,y+1,-z).$$

(Again, cf. [18] and [4]). Indeed, using the projection $(x,y,z)\mapsto y$, it is easy to check that the quotient under Γ yields a T^2-bundle with the desired monodromy. If we choose a function f as in Proposition 5.2 that satisfies the additional condition $f(x)=-f(-x)$, and replace $2\pi y$ by πy in the argument of cos and sin, we obtain a contact circle on $\Gamma\backslash Nil^3$.

This completes the proof of part (ii) of Theorem 5.1.

5.4. The geometry $S^2\times E^1$

There are only two compact, orientable manifolds modelled on the geometry $S^2\times E^1$, namely, $S^2\times S^1$ and $\mathbb{R}P^3\#\mathbb{R}P^3$. We now show that both these manifolds admit a contact sphere, as well as a contact circle that is taut in the neighbourhood of some point.

We think of S^2 as the unit sphere in $\mathbb{R}^3$, so we describe $S^2\times E^1$ in terms of coordinates (x,y,z,t) with $x^2+y^2+z^2=1$.

Clearly, $S^2\times S^1$ is obtained by taking the quotient of $S^2\times E^1$ under a translation in t-direction; $\mathbb{R}P^3\#\mathbb{R}P^3$ is obtained (cf. [16]) by taking the quotient of $S^2\times E^1$ under the maps

$$(x,y,z,t)\longmapsto(-x,-y,-z,2t_i-t),\quad i=1,2,$$

where $t_1\neq t_2$. In other words, these maps are the antipodal map in the S^2-factor and two distinct reflections in the E^1-factor.

Proposition 5.4. *On $S^2\times E^1$, set*

$$\omega_1=x\,dt+y\,d(h(z))-h(z)\,dy,$$

$$\omega_2=y\,dt+h(z)\,dx-x\,d(h(z)),$$

where $h(z)$ is a smooth, monotone increasing function that satisfies

$$\begin{array}{lcll} h(z) & \equiv & 1 & \textit{near } z=1, \\ h(z) & = & z & \textit{near } z=0, \\ h(z) & = & -h(-z) & \textit{for all } z. \end{array}$$

Then (ω_1, ω_2) is a contact circle that is taut near $z = \pm 1$ and descends to the quotients $S^2 \times S^1$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Remark. If we set $h(z) = z$ and $\omega_3 = z\,dt + x\,dy - y\,dx$, we get a contact sphere which also descends to the quotients $S^2 \times S^1$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$. In [6] it was shown that the manifolds that admit a taut contact 2-sphere (defined in analogy to taut contact circles) are precisely the left-quotients of SU(2).

Proof. A straightforward computation yields the following.

$$(x\,dx + y\,dy + z\,dz) \wedge \omega_1 \wedge d\omega_1 = (2x^2 h' + y^2 h' + zh)\,dx \wedge dy \wedge dz \wedge dt.$$

This is a volume form on $(\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^1$, and it is easy to see that ω_1 is invariant under the maps described above, so ω_1 descends to a contact form on $S^2 \times S^1$ and $\mathbb{R}P^3 \# \mathbb{R}P^3$, respectively.

Now define a self-diffeomorphism Φ of $\mathbb{R}^3 \times \mathbb{R}^1$ by

$$\Phi(x, y, z, t) = (\lambda_1 x + \lambda_2 y, -\lambda_2 x + \lambda_1 y, z, t),$$

where $(\lambda_1, \lambda_2) \in S^1$. This diffeomorphism sends $S^2 \times E^1$ to itself and satisfies

$$\Phi^* \omega_1 = \lambda_1 \omega_1 + \lambda_2 \omega_2,$$

thus $\lambda_1 \omega_1 + \lambda_2 \omega_2$ is a contact form on $S^2 \times E^1$ for any $(\lambda_1, \lambda_2) \in S^1$.

Furthermore, Φ^* preserves the forms $x\,dx + y\,dy + z\,dz$ and $dx \wedge dy$, which implies that (ω_1, ω_2) is a taut contact circle near $z = \pm 1$ (where $h' = 0$).

5.5. Connected sums

Let M be a (not necessarily connected) 3-manifold that admits a contact circle (ω_1, ω_2). We now show that it is possible to attach 1-handles near points where the contact circle is taut and extend the contact circle over this 1-handle. In particular, we can form connected sums with all the manifolds described in the preceding sections, thus proving Theorem 5.1 (up to the statement on tightness, which will be proved in the next subsection).

Theorem 5.5. *Let M be a 3-manifold that admits a contact circle (ω_1, ω_2) and assume that this contact circle is taut near two points $P, Q \in M$. Let $\widetilde{M}$ be the manifold obtained from M by 0-surgery along $S^0 = \{P, Q\}$, that is, by removing small discs around P and Q and glueing in a 1-handle. Then $\widetilde{M}$ admits a contact circle $(\widetilde{\omega}_1, \widetilde{\omega}_2)$ that coincides with (ω_1, ω_2) outside a small neighbourhood of the 1-handle.*

Proof. This proof is based on the connected sum construction in [8], which in turn is a generalization of a construction due to Eliashberg [1] and Weinstein [19].

On $\mathbb{R}^4$ with standard coordinates p, q, r, s, consider the 2-forms

$$\Omega_1 = dp \wedge ds + dq \wedge dr,$$

$$\Omega_2 = dq \wedge ds + dr \wedge dp.$$

The vector field

$$\zeta = -2p\partial_p - 2q\partial_q - 2r\partial_r + s\partial_s$$

is what in [8] was called a *generalized Liouville vector field for the* Ω_i, that is, it dilates the summands of the Ω_i by constant factors of equal sign. For instance,

$$\begin{aligned} L_\zeta \Omega_1 &= d(i(\zeta)\Omega_1) \\ &= d(-2p\, ds - 2q\, dr + 2r\, dq - s\, dp) \\ &= -dp \wedge ds - 4\, dq \wedge dr. \end{aligned}$$

It was shown in [8] that if ζ is a generalized Liouville vector field with respect to Ω_1, then $i(\zeta)\Omega_1$ induces a contact form on suitably chosen hypersurfaces transverse to ζ, and this was used to induce so-called hypercontact structures (a quaternionic generalization of contact structures) on a 1-handle, thus allowing to prove a connected sum theorem for hypercontact manifolds.

However, rather than relying on the general theory developed in [8], we give a self-contained construction (closely following [19]) in the low-dimensional setting we are considering here.

In $\mathbb{R}^4$, consider the hyperplanes $\{s=1\}$ and $\{s=-1\}$. Fix an $\epsilon>0$ and let $F(u,v)$ be a smooth function with the following properties:

$$\begin{aligned}
&F(0,0)<0,\\
&\frac{\partial F}{\partial u}\geq 0,\ \frac{\partial F}{\partial u}>0 \text{ for } v=0,\\
&\frac{\partial F}{\partial v}\leq 0,\\
&\left(\frac{\partial F}{\partial u}\right)^2+\left(\frac{\partial F}{\partial v}\right)^2>0,\\
&F(u,1)=0 \text{ for } u>\epsilon^2.
\end{aligned}$$

Then $\widetilde{F}(p,q,r,s)\stackrel{\text{def}}{=}F(p^2+q^2+r^2,s^2)=0$ defines a hypersurface in $\mathbb{R}^4$ which is rotationally symmetric about the s-axis and coincides with the hyperplanes $\{s=\pm 1\}$ for $p^2+q^2+r^2>\epsilon^2$, and thus constitutes a 1-handle between these two hyperplanes, attached in an ϵ-neighbourhood of the points $P_0=(0,0,0,1)$ and $Q_0=(0,0,0,-1)$. We call this a *standard* 1-*handle.*

Lemma 5.6. *Set $\omega_i^0=i(\zeta)\Omega_i$, $i=1,2$. Then (ω_1^0,ω_2^0) induces a contact circle on any standard* 1-*handle. This contact circle is taut on $\{s=\pm 1\}$.*

Proof. Define a self-diffeomorphism Φ of $\mathbb{R}^4$ by

$$\Phi(p,q,r,s)=(\lambda_1 p+\lambda_2 q,-\lambda_2 p+\lambda_1 q,r,s),$$

where $(\lambda_1,\lambda_2)\in S^1$. Note that Φ preserves a standard 1-handle. Then

$$\Phi^*\Omega_1=\lambda_1\Omega_1+\lambda_2\Omega_2$$

and

$$(\Phi_*)^{-1}\zeta=\zeta,$$

which implies

$$\Phi^*\omega_1^0=\lambda_1\omega_1^0+\lambda_2\omega_2^0.$$

Hence, it is enough to show that ω_1^0 induces a contact form on a standard 1-handle.

To this end, we compute

$$d\widetilde{F}\wedge\omega_1^0\wedge d\omega_1^0=\frac{\partial F}{\partial u}(16p^2+4q^2+4r^2)-\frac{\partial F}{\partial v}8s^2,$$

and observe that this expression is always positive on $\{\widetilde{F}(p,q,r,s)=0\}$.

Moreover, on $\{s=1\}$, where we have

$$\omega_1^0 = -dp - 2q\,dr + 2r\,dq,$$

$$\omega_2^0 = -dq - 2r\,dp + 2p\,dr,$$

the contact circle (ω_1^0, ω_2^0) is easily seen to be taut. The same is true on the hyperplane $\{s=-1\}$.

We now use this contact circle on a standard 1-handle to attach a 1-handle to M between the points P and Q. In [6] a Darboux type theorem for taut contact circles was proved, which implies that any two taut contact circles are locally equivalent. This allows to find small neighbourhoods U, V of P and Q in M and neighbourhoods U_0, V_0 of P_0 and Q_0 in $\{s=\pm 1\}$, as well as diffeomorphisms

$$\phi : U \longrightarrow U_0 \ \text{ and } \ \psi : V \longrightarrow V_0$$

that pull back (ω_1^0, ω_2^0) to (ω_1, ω_2). If we now attach a standard handle between P_0 and Q_0 inside the neighbourhoods U_0, V_0, these diffeomorphisms and the contact circle on the standard handle can be used to extend the contact circle (ω_1, ω_2) over a 1-handle between P and Q, attached inside the neighbourhoods U, V.

This concludes the proof of Theorem 5.5

Remarks. (1) The construction of Eliashberg and Weinstein is actually a construction of symplectic handlebodies. This means in particular that if the two 3-manifolds M_1 and M_2 admit symplectically fillable contact structures (see the next section for a definition of *symplectically fillable*), then so does $M_1 \# M_2$. With a little extra care it is possible to show that this is still true in the present context (That is the reason why we work with 1-forms ω_i^0 that are induced by a generalized Liouville vector field ζ from symplectic forms Ω_i on $\mathbb{R}^4$, rather than just write down explicit expressions for the ω_i^0).

(2) In Propositions 5.2, 5.3 and 5.4 we have used an auxiliary function f, g or h, respectively, to obtain contact circles which are taut in some open neighbourhood of a point. As remarked at the end of the preceding section, any contact circle can be deformed locally to a taut contact circle. Thus, the tautness assumption in Theorem 5.5 is redundant.

Observe that if we set $\Omega_3 = dr \wedge ds + dp \wedge dq$ and $\omega_3^0 = i(\zeta)\Omega_3$, then $(\omega_1^0, \omega_2^0, \omega_3^0)$ defines a contact sphere on any standard 1-handle, and this contact sphere is taut on $\{s = \pm 1\}$. Now, there is no Darboux type theorem for taut contact spheres, so Theorem 5.5 does not extend directly to contact spheres. However, the same arguments as in [8] can be used to show that the obvious taut contact sphere on S^3 (cf. [6], [8]) can be deformed locally in such a way that it is possible to glue in a standard 1-handle. This allows to conclude to the following.

Proposition 5.7. *Let M be the connected sum of left-quotients of* $\mathrm{SU}(2)$ *and copies of $S^2 \times S^1$. Then M admits a contact sphere.*

5.6. Tight contact structures

Before proving the statement on tightness in Theorem 5.1, we first recall some definitions and facts from [2], [3], [9].

A contact structure $\mathcal{E} \subset TM$ on a 3-manifold M is called *overtwisted* if there exists an embedded closed 2-disc $D \subset M$ such that ∂D is tangent to $\mathcal{E}$ while the disc D is transverse to $\mathcal{E}$ along ∂D. A non-overtwisted contact structure is called *tight*. The standard contact structures $\{dz + x\,dy = 0\}$ on $\mathbb{R}^3$ and $\{x\,dy - y\,dx + z\,dt - t\,dz = 0\}$ on $S^3 \subset \mathbb{R}^4$ are tight.

The significance of tight contact structures is the following. Whereas the isotopy classification of overtwisted contact structures on a closed 3-manifold coincides with the homotopy classification of tangent 2-plane fields, the classification of tight contact structures is much more subtle. On the one hand, not every homotopy class of 2-plane fields contains a tight contact structure. For instance, on S^3 homotopy classes of tangent 2-plane fields are classified by $\pi_3(S^2) = \mathbb{Z}$, but, up to isotopy, the standard structure is the only tight contact structure on S^3. On the other hand, two tight contact structures which are homotopic as 2-plane fields need not be isotopic as contact structures (such examples exist on T^3).

A contact manifold $(M, \mathcal{E})$ is *symplectically fillable* if there exists a compact symplectic manifold (W, Ω) with $\partial W = M$ and a Liouville vector field X (that is, $L_X\Omega = d(i(X)\Omega) = \Omega$) which is defined near ∂W, everywhere transverse to ∂W, and which is pointing outwards, such that $\mathcal{E} = \ker(i(X)\Omega)$ on $M = \partial W$. (This notion of symplectically fillable is slightly stronger than the one used in [3].)

The main fact that we shall use is that if $(M, \mathcal{E})$ is symplectically fillable, then $\mathcal{E}$ is tight.

First we show that the contact structures we constructed in the preceding sections on the indecomposable manifolds listed in Theorem 5.1 are tight. Note that all the contact structures $\ker(\lambda_1\omega_1+\lambda_2\omega_2)$ in a contact circle are homotopic as contact structures, and hence isotopic, thus it is enough to consider $\mathcal{E}_1=\ker\omega_1$.

The proof in [6] that $\mathcal{E}_1=\ker\omega_1$ is tight if ω_1 is part of a taut contact circle was based on our complete classification of taut contact circles up to homotopy on the manifolds listed in Theorem 1.3. We could show that any such $\mathcal{E}_1$ on a left-quotient of $\mathrm{SU}(2)$ or $\widetilde{\mathrm{E}}_2$ is covered by the tight standard structure on S^3 and $\mathbb{R}^3$, respectively. This clearly implies that $\mathcal{E}_1$ is tight, for any overtwisted disc $D\subset M$ would lift to an overtwisted disc in S^3 or $\mathbb{R}^3$. For the left-quotients of $\widetilde{\mathrm{SL}}_2$ we used the existence of symplectic fillings, which have been constructed in [5].

In case (ii) (where only the Nil^3 and Sol^3 manifolds are left to be considered), we mentioned above that tightness follows from [11]. A more elementary way to see this, however, is to observe that the structures $\mathcal{E}_1$ on Nil^3 and Sol^3, respectively, are diffeomorphic to the tight standard structure on $\mathbb{R}^3$, so again the induced contact structure on any quotient has to be tight as well.

We only show this for Nil^3 and leave the Sol^3 case, as well as the cases in Section 5.3, as exercise. Define a diffeomorphism Φ of $\mathbb{R}^3$ by

$$\begin{aligned}&\Phi(x,y,z)\\ &\quad=\left(x\cos(2\pi y)-z\sin(2\pi y),2\pi y,\left(x+\tfrac{f(x)}{2\pi}\right)\sin(2\pi y)+z\cos(2\pi y)\right),\end{aligned}$$

then one computes that $\Phi^*(dx+z\,dy)=\omega_1$.

Lemma 5.8. *On $S^2\times E^1\subset\mathbb{R}^3\times E^1$ (with coordinates x,y,z,t as in Section 5.4) set*

$$\omega=x\,dt+y\,dz-z\,dy.$$

This descends to a contact form on $S^2\times S^1$, and there it defines a symplectically fillable, and hence tight, contact structure.

Proof. On $B^3\times E^1$ (where B^3 denotes the closed unit 3-ball) we have the symplectic form

$$\Omega=dx\wedge dt+2\,dy\wedge dz.$$

Set

$$X=x\partial_x+\frac{1}{2}y\partial_y+\frac{1}{2}z\partial_z.$$

This is easily seen to be a Liouville vector field with respect to Ω, transverse to $\partial(B^3 \times E^1) = S^2 \times E^1$, and such that $\omega = i(X)\Omega$. Furthermore, everything is invariant under translations in t-direction and hence yields a symplectic filling $(B^3 \times S^1, \Omega)$ of $(S^2 \times S^1, \omega)$.

Remark. It is not difficult to see that the contact structures in the locally taut contact circle in Proposition 5.4, where z is replaced by $h(z)$, also admit symplectic fillings.

It remains to show that tightness is preserved under the forming of connected sums. It is not known whether this is true in general*, but it does hold for the particular manifolds and tight contact structures that we are considering here. It is understood that all connected sums are formed by glueing in standard 1-handles as in Section 5.5.

As remarked after the proof of Theorem 5.5, the connected sum construction is actually a construction of symplectic handlebodies. This implies that the contact structures on the connected sum of a finite number of symplectically fillable contact manifolds is again symplectically fillable, and hence tight. This argument covers $S^2 \times S^1$ and the left-quotients of $\widetilde{\mathrm{SL}}_2$.

The left-quotients of SU(2) are finitely covered by the standard structure on S^3, which is easily seen to be symplectically fillable by the standard symplectic 4-ball. Hence, the connected sum of manifolds of this type (and any number of other symplectically fillable contact manifolds) is finitely covered by a symplectically fillable contact manifold. Thus, we have a tight contact structure on a covering manifold, so the contact structure on the quotient has to be tight.

Remarks. (1) Instead of passing to a finite cover, one can actually use symplectic fillings for the left-quotients of $S^3 = \mathrm{SU}(2)$. The symplectic filling B^4 of S^3 descends to a singular symplectic filling, since the action of a finite subgroup of SU(2) on S^3 extends to an action on B^4 with 0 as fixed point. However, such a singular filling can be deformed to a non-singular filling. This is best seen on the level of holomorphic fillings [3]; the singularities that arise are special cases of the well-known Brieskorn singularities.

(2) $\mathbb{R}P^3$ is a left-quotient of SU(2), so the manifold $\mathbb{R}P^3 \# \mathbb{R}P^3$ in class (iii) is covered by this connected sum argument. Alternatively, one can again use singular symplectic fillings (with removable singularities) as in (1) for the contact structures in the contact circle defined on

*See Note added in proof.

$\mathbb{R}P^3\#\mathbb{R}P^3$ as a quotient of $S^2 \times E^1$.

Finally, we have to deal with the contact manifolds of type (ii), all of which are covered by the tight standard contact structure on $\mathbb{R}^3$. Here we use the fact that the standard contact structure on $\mathbb{R}^3$ is diffeomorphic to the standard contact structure on S^3 with a point removed. From this it follows that forming the connected sum of two copies of the standard $\mathbb{R}^3$ yields a tight contact structure, since it allows a two-point compactification to the connected sum of two copies of the (fillable) standard S^3. The same remains true for any finite number of 1-handles and the connected sum with other symplectically fillable manifolds.

Now, given a connected sum M of manifolds of type (ii) and other symplectically fillable manifolds, we can pass to an infinite cover $\widehat{M}$ which consists only of copies of the standard $\mathbb{R}^3$ and symplectically fillable manifolds, with infinitely many 1-handles between them. However, any embedded (closed) disc $D \subset \widehat{M}$ only meets finitely many of these 1-handles, and the preceding argument shows that such a disc cannot be overtwisted.

This concludes the proof of Theorem 5.1.

Remark. In the course of the proof of Theorem 5.1 we have actually found several ways of constructing contact circles on certain manifolds. For instance, in Section 5.1 we gave two different formulae for contact circles on the left-quotients of Nil^3, and on $\mathbb{R}P^3\#\mathbb{R}P^3$ we obtain a contact circle by either viewing it as a quotient of $S^2 \times E^1$ or a connected sum of two copies of $\mathbb{R}P^3$. Also, if M admits a contact circle, then a contact circle on $M\#(S^2 \times S^1)$ can be obtained either by attaching a 1-handle to M or by forming the connected sum with $S^2 \times S^1$. This naturally raises the question of equivalence of contact circles. For *taut* contact circles these classification questions have been discussed at great length in [6].

6. An overtwisted contact sphere

In this section we construct an immersion $i : \mathbb{R}^3 \to \mathbb{R}^3$ such that the standard (tight) contact structure on $\mathbb{R}^3$ pulls back to an overtwisted contact structure. By the same map (composed with a self-diffeomorphism of $\mathbb{R}^3$) we can pull back the standard contact sphere on $\mathbb{R}^3$, and this proves the following.

Proposition 6.1. *There is a contact sphere* $(\omega_1', \omega_2', \omega_3')$ *on* $\mathbb{R}^3$ *such* *that* $\ker \omega_3'$ *is an overtwisted contact structure.*

Consider the contact form $\omega = dz - x\,dy$ defining the standard contact structure on $\mathbb{R}^3$. Consider also the following contact sphere on $\mathbb{R}^3$:

$$\begin{aligned}\omega_1 &= dx - y\,dz + z\,dy,\\ \omega_2 &= dy - z\,dx + x\,dz,\\ \omega_3 &= dz - x\,dy + y\,dx.\end{aligned}$$

which we call the *standard contact sphere* on $\mathbb{R}^3$. We observe that $\omega = \Phi^*\omega_3$, where Φ is the following global diffeomorphism of $\mathbb{R}^3$:

$$\Phi(x,y,z) = \left(\frac{x}{2}, y, z - \frac{xy}{2}\right).$$

Therefore if an immersion $i : \mathbb{R}^3 \to \mathbb{R}^3$ pulls ω back to an overtwisted contact form, then the triple $(\omega_1', \omega_2', \omega_3')$ given by $\omega_j' = i^*\Phi^*\omega_j$, $j = 1,2,3$, is a contact sphere and ω_3' is overtwisted. Clearly this sphere contains many contact circles with an overtwisted contact form.

In order to find the desired immersion i, we first construct an immersed overtwisted disc. Let $i_0 : S^2 \to \mathbb{R}^3$ be an immersion such that the image $i_0(S^2)$ has transverse self-intersection precisely along one simple closed curve and such that the restriction of i_0 to the equator $S^1 \subset S^2$ is an immersion into the xy-plane with exactly two transverse self-intersections and total enclosed area (with sign and with respect to the area form $dx \wedge dy$) equal to zero. Furthermore, choose the immersion i_0 in such a way that a neighbourhood of the equator maps onto a vertical cylinder over the curve $i_0(S^1)$. If this cylindrical part is chosen tall enough, then the characteristic foliation $T(i_0(S^2)) \cap \ker \omega$ has a closed leaf. In other words, if $t \mapsto (x(t), y(t))$, $0 \le t \le 2\pi$, is a parametrization of the immersion $i_0|S^1$, this lifts to a Legendre embedding of S^1 defined by $t \mapsto (x(t), y(t), z(t))$ with $z(0) = 0$, say, and $\dot{z}(t) = x(t)\dot{y}(t)$; and we require that this lifted curve lie on $i_0(S^2)$.

This closed leaf separates $i_0(S^2)$ into two immersed overtwisted discs in $\mathbb{R}^3$ for ω, since ∂_z is tangent to $i_0(S^2)$ and transverse to $\ker \omega$ along the embedded Legendre curve $i_0(S^1)$. Now extend the immersion of one of these discs D^2 to an immersion $i : U \cong \mathbb{R}^3 \to \mathbb{R}^3$ of an open neighbourhood U (diffeomorphic to $\mathbb{R}^3$) of D^2 in $\mathbb{R}^3$. Then $i^*\omega$ defines a contact structure on $U \cong \mathbb{R}^3$ that contains an embedded overtwisted disc, hence an overtwisted contact structure.

A concrete example for such an immersion $i_0 : D^2 \to \mathbb{R}^3$ can be given as follows. Let (r,t) be polar coordinates in $\mathbb{R}^2$ and identify D^2 with the set

$$\left\{0 \leq t \leq 2\pi,\ 0 \leq r \leq 3 + \cos t \sin^3 t\right\}.$$

Then define

$$i_0(r,t) = (a(r)[b(r)\sin t + (1-b(r))\sin(3t)], a(r)\cos t, 3 - c(r)),$$

where $a(r), b(r), c(r)$ are smooth functions satisfying the following conditions, with δ a small positive real number:

(i) $a(r)$ is monotone increasing, $a(r) = r$ for $r \leq \delta$, $a(r) \equiv 1$ for $r \geq 1$.

(ii) $b(r)$ is monotone decreasing, $b(r) \equiv 1$ for $r \leq \delta$, $b(r) \equiv 0$ for $r \geq 1$.

(iii) $c(r)$ is monotone increasing, $c(r) \equiv 0$ for $r \leq \delta$, $c'(r) > 0$ for $r > \delta$, $c(r) = r$ for $r \geq 1$.

Fig. 6 shows some horizontal slices of the image of this immersion. Along the boundary ∂D^2, this immersion restricts to the Legendre embedding of S^1 given by

$$t \longmapsto (\sin(3t), \cos t, -\cos t \sin^3 t),\quad 0 \leq t \leq 2\pi,$$

and because of $c'(r) > 0$ along ∂D^2 the immersion $i_0 : D^2 \to \mathbb{R}^3$ is transverse to $\ker\omega$ along ∂D^2.

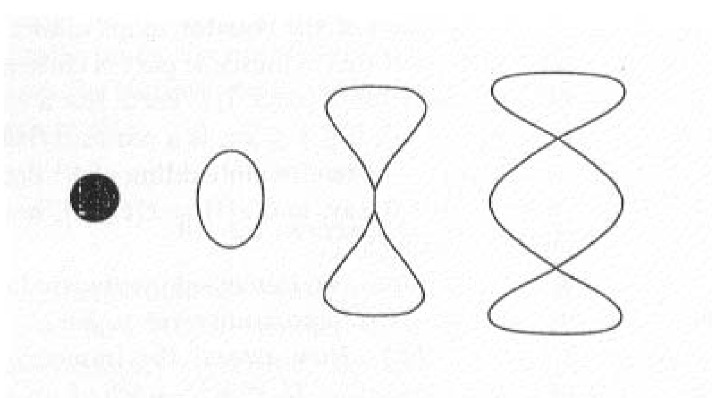

FIGURE 6. Horizontal slices of the immersed disc $i_0(D^2)$.

Acknowledgements

We are most grateful to R. Gompf for many fruitful discussions, and thank J. Cilleruelo for the concise proof of Lemma 2.7.

Note added in proof. It has been shown by V. Colin and, independently, S. Makar-Limanov, that it is indeed possible to perform 0-surgery within the class of tight contact 3-manifolds. See V. Colin, *Chirurgies d'indice un et isotopies de sphères dans les variétés de contact tendues,* C. R. Acad. Sci. Paris Sér. I **324** (1997) 659–663.

References

[1] Y. Eliashberg, *Topological characterization of Stein manifolds of dimension* > 2, Internat. J. Math. **1** (1990) 29–46.

[2] ______, *Contact 3-manifolds twenty years since J. Martinet's work*, Ann. Inst. Fourier (Grenoble) **42** (1-2) (1992) 165–192.

[3] Y. Eliashberg & M. Gromov, *Convex symplectic manifolds*, Proc. Sympos. Pure Math., Amer. Math. Soc. **52** (2) (1991) 135–162.

[4] H. Geiges, *Symplectic structures on* T^2*-bundles over* T^2, Duke Math. J. **67** (1992) 539–555.

[5] ______, *Symplectic manifolds with disconnected boundary of contact type*, Internat. Math. Res. Notices (1994) 23–30.

[6] H. Geiges & J. Gonzalo, *Contact geometry and complex surfaces*, Invent. Math. **121** (1995) 147–209.

[7] ______, *Topology and analysis of contact circles*, Contact and Symplectic Geometry (C.B. Thomas, ed.), Cambridge Univ. Press, 1996, 66–77.

[8] H. Geiges & C.B. Thomas, *Hypercontact manifolds*, J. London Math. Soc. (2) **51** (1995) 342–352.

[9] E. Giroux, *Topologie de contact en dimension* 3 (*autour des travaux de Yakov Eliashberg*), Sém. Bourbaki, Vol. 1992/93; Astérisque **216** (1993) 7–33.

[10] J. Gonzalo, *Branched covers and contact structures*, Proc. Amer. Math. Soc. **101** (1987) 347–352.

[11] H. Hofer, *Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three*, Invent. Math. **114** (1993) 515–563.

[12] W.B.R. Lickorish, *A representation of orientable combinatorial* 3*-manifolds*, Ann. of Math. (2) **76** (1962) 531–540.

[13] J. Martinet, *Formes de contact sur les variétés de dimension* 3, Proc. Liverpool Singularities Sympos. II (C.T.C. Wall, ed.), Lecture Notes in Math. Vol. 209, Springer, Berlin, 1971, 142–163.

[14] D. Rolfsen, *Knots and links*, Publish or Perish, Houston, 1976.

[15] K. Sakamoto & S. Fukuhara, *Classification of T^2-bundles over T^2*, Tokyo J. Math. **6** (1983) 311–326.

[16] P. Scott, *The geometries of* 3*-manifolds*, Bull. London Math. Soc. **15** (1983) 401–487.

[17] W.P. Thurston & H.E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975) 345–347.

[18] M. Ue, *Geometric* 4*-manifolds in the sense of Thurston and Seifert* 4*-manifolds* I, J. Math. Soc. Japan **42** (1990) 511–540.

[19] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. **20** (1991) 241–251.

Eidgen. Tech. Hochschule, Zürich
Universidad Autónoma de Madrid